DOCUMENT RESUME

ED 162 875 SE 025 375

AUTHOR	Dilworth, R. P.; And Others
TITLE	Studies in Mathematics, Volume XVIII: Puzzle Problems and Games Project. Final Report.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	68
NOTE	218p.; For related documents, see SE 025 371-374 and ED 143 544-557; Not available in hard copy due to marginal legibility of original document
EDRS PRICE	MF-$0.83 Plus Postage. HC Not Available from EDRS.
DESCRIPTORS	Curriculum; *Discovery Learning; *Elementary School Mathematics; Elementary Secondary Education; *Games; Instruction; *Instructional Materials; Mathematics Education; *Secondary School Mathematics; *Teaching Guides
IDENTIFIERS	School Mathematics Study Group

ABSTRACT

This is a self-contained manual for use by teachers in preparation for classroom presentations. One of the goals of the report is to show how games and puzzles can provide effective means for developing mathematical understanding and skills. The authors indicate that this type of activity is well adapted for discovery teaching techniques. The report is organized into two main parts. The first part contains experimental units that were tested in the classroom. The topics in this part include: (1) nim-type games; (2) polyominoes; (3) symmetry; (4) a counting machine; (5) finding the greatest common divisor; (6) linear function games; and (7) games with addition tables. The second part consists of the report of a project to compile a list of games and puzzles appropriate for use in the mathematics classroom. Twenty-seven papers contain (in addition to the above list) activities such as: (1) magic squares; (2) Fibonacci problems; (3) geometric puzzles; (4) numerical oddities; and (5) powers and primes. (MP)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

SCHOOL
MATHEMATICS
STUDY GROUP

U.S. DEPARTMENT OF HEALTH,
EDUCATION & WELFARE
NATIONAL INSTITUTE OF
EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY

SMSG

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC) AND USERS OF THE ERIC SYSTEM."

ED162875

Studies in Mathematics

VOLUME XVIII

Puzzle Problems and Games Project
Final Report

SE 025 375

2

SMSG

Studies in Mathematics

Volume XVIII

Puzzle Problems and Games Project

Final Report

The following is a list of all those who participated in the preparation of this volume:

R. P. Dilworth, Calif. Institute of Technology, Pasadena
Shirley Hill, University of Missouri
James Henkelman, University of Maryland
Walter Jacobs, National Security Agency
Joseph Latino, Northport School System, New York
W. G. Lister, State University of New York
Brockway McMillan, Summit School System, N. J.
Frank Sinden, Bell Telephone Labs
C. J. Titus, University of Michigan
Herbert Wills, University of Illinois
Robert Wirtz, Carmel, California

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

Permission to make verbatim use of material in this book must be secured from the Director of SMSG. Such permission will be granted except in unusual circumstances. Publications incorporating SMSG materials must include both an acknowledgment of the SMSG copyright (Yale University or Stanford University, as the case may be) and a disclaimer of SMSG endorsement. Exclusive license will not be granted save in exceptional circumstances, and then only by specific action of the Advisory Board of SMSG.

© 1968 by The Board of Trustees of the Leland Stanford Junior University
All rights reserved
Printed in the United States of America

TABLE OF CONTENTS

Introduction 1

Appendix A

Puzzle Problems and Games Project - Writing Teams 9

Appendix B

Puzzle Problems and Games Project - Participating Teachers and Classes 10

Appendix C

Experimental Units 11

Nim-Type Games 13

Polyominoes 21

Symmetry 45

Classroom Activities Based on Computer Concepts a Counting Machine 59

Classroom Activities Based on Computer Concepts Finding the Greatest Common Divisor 69

Linear Function Games 79

Games With Addition Tables 97

Games With Multiplication Tables 104

Appendix D

Report of the Conference for the Development of Mathematical Puzzle-Problems and Games 111

Introduction 112

Table of Contents for Appendix D 119

PUZZLE PROBLEMS AND GAMES PROJECT

FINAL REPORT

INTRODUCTION

This is the final report on the Puzzle Problems and Games Project sponsored by the School Mathematics Study Group through funds provided by the National Science Foundation.

The impetus for the project came from the conviction of a number of mathematicians and mathematics educators that games and puzzles can provide a particularly effective means for developing mathematical understanding and skills. However, not every mathematical recreation has significant educational value. Hence, a first objective of the project was to survey the literature on mathematical recreations for those games and puzzles which would be appropriate for use in the mathematics classroom. This was carried out at a two-week conference held at Stanford University in June 1965. Seven mathematicians with particular interest in educational applications of mathematical recreations participated in this conference. In addition to a survey of the literature, several brainstorming sessions were held with the aim of developing new ideas for mathematical games and puzzles. The report of the conference, consisting of 27 working papers, is incorporated as Appendix D of this report.

Formulating a game or puzzle idea with solid educational potential is only the first step in making it available for classroom use. The idea must be presented to the teacher in such a way that he can arouse the students' interest and insure that the educational features of the game or puzzle problem is not lost. If such a presentation requires a special training session for the teacher, then the usefulness of the particular game or puzzle will be severely restricted. Hence, the second objective of the project was to prepare some experimental materials of the games and puzzles which were self-contained and which would not require special training sessions in order to be used effectively by teachers.

In order to implement this second phase of the project, four teams, each consisting of a mathematician and an educator, were organized. Each team was charged with the task of developing appropriate written material from one or two of the more promising ideas collected in the first phase of the project. This writing effort was carried out during the winter and spring of 1966-67. In June 1967, the four teams met in Stanford for a one-week tryout and evaluation session. Twelve teachers with their respective classes volunteered for the experiment. Each class was assigned one of the topics developed by the writing teams. In order to insure adequate preparation time, the teachers received the written material several days prior to the scheduled class sessions. Each teacher presented the material to the class during a regularly scheduled class period with the members of two or more of the writing teams acting as observers. On the basis of these observations the written material was evaluated and, in some cases, appropriate modifications were made. The members of the writing teams and the list of the participating teachers are given in Appendices A and B. Appendix C contains the modified drafts of the material prepared by the four teams.

CONCLUSIONS AND RECOMMENDATIONS

With regard to the project as a whole, it was the concensus of the participants that the format adopted in this project is well suited to the task of producing such experimental material. The brainstorming sessions seem to work well for the conceptual phase of the project while the mathematician-educator team approach had many advantages in the writing phase.

It was noted by several participants, that although games and puzzle activities may be used for several years in succeeding classes, eventually the ideas get passed around among the pupils and the novelty of the idea is lost. Hence, there is a continuing need for the discovery of new ideas for games and puzzle problems which have significant educational merit. Such ideas are not easily produced as the participants in the first phase of the project quickly discovered. In fact, producing a new idea of this kind is not unlike the problem of producing a new idea in a piece of mathematics research. Accordingly, it seems to be clear that research mathematicians are the appropriate persons to involve in such idea-generating activities. It is recommended that a serious effort be made to interest research mathematicians in devoting some of their talents to this task. Perhaps every other year a small group of these individuals could be gathered together to spend a two-week period searching for new ideas in games and puzzles with significant mathematical content. A program of this kind might well provide the continuing supply of ideas for mathematical games and puzzles which will keep these activities fresh and challenging to students. The participants in the June 1965 conference noted that the two-week period is about maximal, since after that time the wells of inspiration begin to run dry.

The writing teams, consisting of a mathematician and a mathematics educator, seemed to work out very well. The mathematician is responsible for the mathematical content of the material and must insure that the mathematical ideas are correctly presented and are not obscured or distorted by the recreational aspects of the activity. On the other hand, it is important for the educator to make judgments concerning the capabilities of teachers and students. This will include such questions as the choice of the method for presenting the material to the teacher, the level of the language used and the depth at which the material is treated.

It is the recommendation of the participants that the School Mathematics Study Group sponsor a continuing program of development of written material similar to the experimental units produced in this project. This does not need to be organized on a large scale. For example, if SMSG could support one or two mathematician-educator writing teams each summer, it would in a short time provide an abundant source of games and puzzle problem material suitable for teachers. These teams could be either new teams organized for the summer in question or teams who have already demonstrated their ability to produce successful material and are being encouraged to continue their work. In either case, it is highly important that there be classes available for classroom tryout as the written material is being produced. The experience of writing teams in this project showed clearly that it was almost impossible to anticipate the difficulties which may arise in the use of experimental material of this kind. It should be pointed out that there are many ideas collected in the first phase of the project which still need to be developed into a suitable written form for classroom use.

We turn now to some conclusions and recommendations concerning the units produced by the writing teams.

The principal conclusion from the experimental use of the material in the classroom is that games and puzzle problem activities are effective only if the teacher has thoroughly mastered the material before presenting it in the classroom. In particular, the teacher should have devoted enough time to personally playing the game or working with the puzzle problem that most of the difficulties or ramifications which may occur in the classroom will have been anticipated. It is also highly important that the teacher understand the mathematical principles involved and be very much aware of the educational value of the activity.

Games and puzzle problems are particularly well adapted for discovery teaching techniques. One of the principal educational advantages of these activities is the fact that the basic mathematical principles and related skills are learned as a natural by-product of playing the game or seeking the solution of the puzzle problem. In some games, for example the Nim-type games, the discovery of the mathematical principle is the whole point of the game. The game would indeed be useless if the principle were explained at the start. Hence, the discovery approach is absolutely essential in presenting such games. Furthermore, it is highly important for the teacher to maintain a flexible attitude in class, since different pupils may use quite different approaches to uncovering the principles involved.

Next, it should be emphasized that good classroom management is essential if the educational potentials of games are to be realized. In particular, the rules must be clearly formulated, the precise boundary conditions must be determined, and the rules must be fair.

Some specific conclusions concerning the individual units are the following:

(1) Polyominoes

The material in this unit appeared to go well in both grades 3 and 5. The fifth grade children were able to proceed more rapidly and to do more problems than the third grade children.

It is most important that the teacher construct a set of pentaminoes and to have worked through the material using these configurations before presenting the material to the class. The teacher should also thoroughly understand the basic symmetry principles and the symmetry properties of the pentaminoes before discussing this subject with the class. It was found that even young children have a very good intuitive feeling for the properties of these configurations.

(2) Nim-type games

These games provide an unusual opportunity for the student-players to discover winning strategies based upon mathematical principles. One teacher commented that there was more analytical thinking taking place in the one-hour class period devoted to these games than had occurred in months of the regular mathematics class. Furthermore, several pupils in this fifth grade class who had shown only cursory interest in the regular mathematics classes suddenly came alive and exhibited a high degree of analytical skill while participating in the game.

Nim-type games are basically two-person games. When they are adapted to full class participation, great care must be exercised to see the game is well organized, that all of the students have a stake in the outcome of the play, that each student has a chance to participate and all are treated fairly. One such organization is suggested in the write-up of the unit.

(3) Symmetry

This unit places unusual demands upon the teacher since the material is very different from that normally treated in the elementary mathematics curriculum. Thus it is particularly important in this case, that the teacher work through all of the questions and problems in detail and study the shapes thoroughly.

Outlines of the configurations into which the shapes are to be fitted should be drawn beforehand. It is distracting and time consuming to do it during the discussion.

The mirror is a very valuable prop, especially in showing up errors on the worksheets. When an N turns into an M the effect is dramatic and convincing.

The parallelogram is a good shape for distinguishing rotational from reflective symmetry. Many children (and adults) have to do quite a bit of folding before being convinced that the parallelogram has no line of symmetry.

It should be noted that, with respect to this material, asking a child to do something impossible may be a good instructional device. Children do not like to accept the impossible and will exert a great deal of mental effort in an attempt to get around the impossibility.

(4) A Counting Machine

The use of students to imitate the components of a simple computing machine affords a very satisfactory method of introducing the basic computer ideas in the early grades. The example described in this unit works well in grades 2 - 5.

When a student takes the part of a machine which can be in one of two states, it is highly important that the two positions of the student be very distinct. Sitting versus standing appears to work well since it is difficult to maintain a position half way between. Raising and lowering of a hand is not good, since a raised hand can gradually slip down.

In this activity it is important for the students taking the parts of the machine to be changed frequently. This enables the teacher to take advantage of classroom participation. Furthermore, the students are always eager to try to correct the wrong operation of a part of the machine.

(5) Finding the Greatest Common Divisor

This activity, which illustrates the algorithmic approach to mathematics, is probably too advanced for the elementary grades. It could well be used at the junior high level.

Since most teachers will not be familiar with the greatest common divisor algorithm, the teacher must devote sufficient time beforehand to thoroughly understand the process.

(6) Linear Function Games

The objective of these games is to discover the individual components of a complex mathematical operation by observing what the operation does to a variety of inputs. Thus these games are extremely well adapted to the use of discovery-teaching methods. In fact, it is essential that the discovery approach be used, since if the basic facts are given away or if the students are given too many hints, the main point of the game is lost. The students must be given every opportunity to make poor as well as good choices for inputs and to discover for themselves the extent to which the basic components are determined by input-output results.

The teacher can, however, contribute significantly to the educational effectiveness of the games by asking, at appropriate times, such questions as, "How do you know that ----?" or "What will happen if ----?"

These games are, by nature, open-ended and cannot be conducted in a purely mechanical way.

(7) Games with Addition Tables

These games have many of the same features as the linear function games and the comments made with respect to those games apply to these as well.

The participants discussed several possibilities concerning the further use and/or development of the written units prepared during the project. They recommend that several of these units be printed up in pamphlet form with a forward to the teacher covering the main points noted above concerning the use of the material. These pamphlets should be given a reasonably wide distribution to interested teachers with appropriate provisions for feedback. The results should give some definitive information on (1) the interest of teachers in using such material in their classes and (2) the effectiveness of the material when no special motivational activities are provided for the teachers.

Finally, the participants wish to express their appreciation to the School Mathematics Study Group for making the project possible and for providing the necessary facilities for each phase of the undertaking. We are particularly grateful for the enthusiastic cooperation of teachers and their classes in the tryout phase of the project.

R. P. Dilworth

APPENDIX A

PUZZLE PROBLEMS AND GAMES PROJECT

Writing Teams

Professor R. P. Dilworth
Calif. Inst. of Tech.
Pasadena, Calif.

Dr. Walter Jacobs
National Security Agency
Ft. George G. Meade, Maryland

Professor W. G. Lister
State University of New York
Stony Brook, New York

Dr. Frank Sinden
Bell Telephone Labs.
Murrey Hill, New Jersey

Professor Shirley Hill
University of Missouri
Kansas City, Missouri

Professor James Henkelman
University of Maryland
College Park, Maryland

Mr. Joseph Latino
Northport School System
Northport, New York

Mrs. Brockway McMillan
Summit School System
Summit, New Jersey

Professor Henry Walbesser of the University of Maryland worked with Dr. Jacobs and Prof. Henkelman during the early stages of the writing project and contributed many useful suggestions.

Due to illness, Mrs. McMillan was unable to participate in the tryout sessions. However, she contributed many valuable comments and criticisms during the writing phase of the project.

APPENDIX B

PUZZLE PROBLEMS AND GAMES PROJECT

Participating Teachers and Classes

Mr. Leslie Apay
Grade 5
Fairmeadow, Palo Alto

Mr. Paul Erickson
Grade 6
Fairmeadow, Palo Alto

Mrs. Evelyne Gritter
Grade 4
Garland, Palo Alto

Miss Carol Hoge
Grade 3/4
Fairmeadow, Palo Alto

Mrs. Mary Mathias
Grade
El Carmelo, Palo Alto

Miss Carol Mitchell
Grade
Garland, Palo Alto

Miss Dagmar Nannfeldt
Grade 5
Hoover, Palo Alto

Mr. Vernon Pike
Grade 5
Hoover, Palo Alto

Mr. Richard Spinelli
Grade 5/6
El Carmelo, Palo Alto

Mrs. Beverly Wik
Grade 4
Addison, Palo Alto

APPENDIX C

Experimental Units

Nim-type Games	R.P. Dilworth and S. Hill
Polyominoes	R.P. Dilworth and S. Hill
Symmetry	F. Sinden
A Counting Machine	W. Jacobs and J. Henkelman
Finding the Greatest Common Divisor	W. Jacobs and J. Henkelman
Linear Function Games	W.G. Lister and J. Latino
Games with Addition Tables	W.G. Lister and J. Latino

NIM-TYPE GAMES

(Prepared by R. P. Dilworth and S. Hill)

INTRODUCTION

Nim-type games are played by two players and may be described as follows: A starting sequence of numbers is given; the players play alternately, each play consisting of a modification of the sequence of numbers within limits prescribed by the rules of the game. The first player to change the sequence into one of a given set of target configurations wins the game.

Perhaps the simplest Nim-type game is that in which the starting sequence consists of the single number zero. Each play consists of adding one of the numbers 1, 2 to the number previously obtained. The first player to reach a given target number, for example 7, wins.

A slightly more complicated game of this type is the game of "21". Again the starting sequence consists of the single number zero. Each play consists of adding one of the numbers 1, 2, 3, 4, 5 to the number already obtained. The target number in this case is the number 21.

In the classical game of Nim, the starting sequence consists of the three numbers 3, 5, 7. Each play consists of replacing any one of the numbers in the sequence by a smaller whole number. The first player to obtain the sequence 0, 0, 0 wins the game.

The features of the previous two games may be combined as follows: The starting sequence consists of the three numbers 0, 0, 0. Each play consists of adding 1, 2, 3, 4, or 5 to one of the numbers of the sequence. The first player to obtain a sequence containing the number 21 wins.

As a final example, it should be observed that the game need not be played with numbers. Let us suppose that the players are provided with a selection of rods of various lengths. The first player selects a rod and places it on the playing field. The second player then selects a rod and places it on the playing field so that it extends the rod already played in a straight line. The players continue to play in this manner. The player wins who first makes the combined lengths of the rods exceed a certain target length.

These examples should be sufficient to indicate the wide variety of possible Nim-type games.

Experience has shown that games of this type are capable of stimulating the interest of persons of nearly every age group. However, we are not principally interested in the entertainment value of the game, but rather in ways in which they can contribute to mathematical skills and understanding.

Clearly Nim-type games can be used as an exercise in elementary addition and subtraction. As such, they are probably applicable only to the early elementary grades. Increasing the target numbers and enlarging the selection set, while complicating the arithmetic, does not add appreciably to the interest of the game. Indeed, by greatly increasing the number of possible strategies, it may simply make the game more confusing to the players. However, the game can always be tailored to fit the level of arithmetical skill of the players.

By far, the most significant mathematical contributions of Nim-type games lie in the area of problem solving. Namely, the development of a strategy of play leads to superb examples of problems in which elementary but non-trivial mathematical thinking is needed in reaching solutions. The approaches which these problems require are exactly those which occur in many areas of mathematics. Furthermore, the rewards for making progress toward solutions are real and immediate -- the player wins more games. In fact, a complete solution of the strategy problem means that the player can win every game he plays.

The only immediate return to the pupil who solves a text book problem is either the teacher's approval or the personal satisfaction of obtaining a solution. Neither of these rewards may be sufficient to motivate the student to make the effort required in problem solving. On the other hand, the challenge of competitive play places a great premium on the development of a winning strategy, and hence provides a very strong motivation for solving the strategy problems.

Since the solutions of even simple strategy problems involve a little mathematical maturity, it would not be appropriate to consider this aspect of Nim games in the early elementary grades. However, there is every reason to expect that 5th or 6th grade students are capable of discovering winning strategies for these games.

CLASSROOM GAMES AND ANALYSIS

Nim-type games are two-player games. Thus classroom use could be confined to occasions when two students can sit down apart from their classmates, learn the rules of a game and play it. The teacher's role is to teach the rules, use judgment as to when to suggest a variation or new game, and perhaps to discuss (but not reveal) strategy with the players.

It may be advisable, at some point, to play some of the games as team games, either with the class divided into two large teams and players taking turns calling out a play, or with two small groups as teams consulting on their plays and the rest of the class as spectators. The obvious advantage of the former is that more children actively participate but the disadvantage is that one cannot really develop a strategy unless one can decide every play. The small-group teams are a compromise, wherein the members discuss and agree upon their plays and their strategy.

An example of organization which gives the entire class a chance to participate is the following:

The class is divided into two teams. Each team consists of pairs of players who work together. The game is made up of a series of matches between pairs, one pair from each team. For one game, each pair plays one match and one point is scored by the pair which wins the match. At the end of all matches the team with the most points wins.

For example, let us suppose the game is "14" with a selection set of {1,2}. The first pair on team A plays against the first pair on team B. The winners score a point for their team. In the meantime, all other players are trying to work out the strategy so they benefit from attending closely to the play. Then the second pair from team A plays against the second pair from team B in the same game, "14". This continues until all pairs have played a match. The rules are that team members can only talk to their partners during the entire game. They cannot discuss strategy with other players on either team.

After all matches are completed, an extra point can be scored by the team which did not go first if the captain or someone he designates can explain a winning strategy and show it is a winning strategy by playing a sample game using that strategy.

GAME OF "7"

It appears to be highly desirable to begin with a very simple, but nontrivial game of Nim-type. Such a game would be the following: Let the starting number be zero, choose 7 as the target number and let the selection set be the numbers 1 and 2. Thus the first player chooses one of the numbers 1 or 2. The second player likewise chooses one of the numbers 1 or 2 and adds it to the number selected by the first player. The first player then again selects 1 or 2 and adds it to the sum obtained by the second player. They continue in this manner until one of the players makes the sum equal to 7. This player is the winner.

It is important for the students to have some experience playing the game and making decisions since this is the only way they will discover that there is indeed a problem to be solved in winning the game and thus be motivated to try to find a solution. The first prerequisite in understanding a game is acutal experience in playing the game. This is important for the teacher since without this experience it will be difficult to appreciate the pupils' choice of a method of play and their approach to an understanding of the game.

Since this is a fairly simple game, some pupils may be able to discover a winning strategy after playing relatively few games. These pupils should be warned not to give away the winning strategy to the other pupils. On the other hand, the discovery of a winning strategy by a few of the students may be a challenge for other pupils to do the same.

GAME OF "4"

If many of the students fail to find quickly a winning strategy for themselves, the teacher is presented with an excellent opportunity to illustrate one of the basic principles of problem solving: _If a problem does not yield to a direct attack, try to solve a simpler problem which has many of the features of the given problem_. In this case, the teacher may suggest that the pupils try the game in which the target number is made smaller, say 4. For the new game, the pupils should quickly discover that choosing 2 is a poor first move since the second player can then choose 2 and win. Hence, the best first move is to choose 1. Then regardless of the choice of the second player, the first player can always make the sum equal to 4. Having found the winning strategy for the simpler game, the pupils should now be able to discover the proper strategy for the game with the target number 7. In fact, since they have found that it is always possible for the first player to make the sum equal to 4, it is an easy step to see that regardless of what number the second player chooses next, the first player can make the sum equal to 7 and thus win. It follows that the winning first move is to choose the number 1.

It may be that some pupils may discover that the choice of 1 is the best first move without understanding the rest of the strategy. It is important that they continue until they clearly realize that the winning play consists of choosing 1 on the first play and then making a sum of 4 and finally 7.

It is also worthwhile for the students to discover that if the first player at any time makes an incorrect play, the second player can always go on to win. When the students have pretty well mastered this simple game, it is probably the appropriate time to propose the problem: How does the winning play change if the target number is changed to 8? This question should test whether they really understand the principle of the winning strategy. If they do, then they should be able to decide quickly what changes should be made in the winning strategy. The teacher may also try the target numbers 9, 10, and 11 to further reinforce the understanding. Finally, the value of a general description of the winning strategy can be emphasized by setting a very high target number, say 37. Here the pupils may discover the important principle that the correct first move is the remainder obtained by dividing the target number by 3.

GAMES WITH SELECTION SETS OF {1,2,3}

After the solution to the game with the selection set {1,2} is clearly understood, the next natural step is to change the game in another direction, namely, change the selection set. This method for developing general techniques for problem solving is a standard approach in all of mathematics. This may be done in easy stages by raising the target number to 14 and changing the selection set to {1,2,3}, or perhaps one can go directly to the game "21" mentioned in the introduction. Again, the pupils should begin by playing the game under the new conditions. As they try to develop a winning strategy, the pupils should be encouraged to make use of the results which they have obtained for the simpler game and to try to generalize them to the new situation. They should also be encouraged to make conjectures and to test them in actual play or by analysis. Again it should be noted that this approach is basic in all problem solving in mathematics.

In the "14" game, the object of activity is for the pupils to discover for themselves that the correct first choice is the number 2 and that the numbers which the first player must obtain during play exceed 2 by a multiple of 4. Likewise, for the "21" game, the object is to discover that the correct first choice is 3 and the winning number which the first player must obtain exceeds 3 by a multiple of 6.

20

For example, by trying the number 1 as a beginning, he may lose but in the play he may notice that 2 would have been a better move because he notices a pattern in the results. In the next play he tests this to see if his conjecture about patterns works. Such a conclusion may then apply in principle to other variations. This discovery of a strategy may be entirely individual or it may be discussed by a team of players.

There is probably a value in asking someone who has discovered a winning strategy to give a convincing argument as to why it works. In this way he will be making the first steps in formulating a mathematical proof. The teacher's judgment must determine when this is appropriate and to whom the strategy is revealed. For example, it will be of more benefit to other students to continue playing and trying to discover a strategy for themselves. The child who discovers and verifies a strategy in play may just discuss his argument with the teacher and then leave the game to the others. On the other hand, a team effort in jointly conjecturing and testing may be valuable experience. The main objective is that the achievement of a strategy of play is a product of each child's effort and not just what someone else tells him.

OTHER MODIFICATIONS

From this point on, there are a wide variety of modifications of the game which lead to new problems in determining the winning strategy.

SUBTRACTION GAMES

The game can be changed to a subtraction game by setting the beginning number to be a positive whole number and, in each play, one of the numbers in the selection set is subtracted. The target number in this case is zero.

OTHER SELECTION SETS

Instead of taking the selection set to be a consecutive set of integers beginning with 1, one may take a consecutive set beginning with another number, for example, the set {3,4,5}. If the beginning number is zero and the target number is 21, the correct first choice is 5 and the winning numbers for the first player exceed 5 by a multiple of 8. It should be noted that this game may end in a draw if the winning strategy is now followed. For example, if the sum 20 is obtained, then it is impossible to get the sum 21. Also note that if the target number is 22 then it is impossible for the first player to insure a win although he can insure a draw by choosing 4 or 5 in the first play. Again it is hoped that the players will discover these facts for themselves. A very good strategist at this point might be able to describe a general winning strategy for any consecutive selection set and appropriate target numbers.

If the selection set does not consist of consecutive numbers, say 2, 5, entirely new problems arise in analyzing the game.

TWO-NUMBER BEGINNINGS

The next natural complication is to make the beginning sequence consist of two numbers say, for example (0,0). Each play consists of choosing one of the numbers 1, 2 and adding it to one of the numbers in the sequence. Hence, there are two choices; the choice of the number from the selection set and the choice of the term of the sequence to which the number is to be added. The player wins who produces a sequence containing the number 7. The analysis of this game shows that the first player cannot insure a win. Hence, it is better to use the sequence (0,1) as a starter. The correct first play is to add 1 to the first number of the sequence producing the sequence (1,1). The other winning sequences are (1,4), (4,1), and (4,4). It should be noted that the solution in this case is closely related to the solution of the corresponding game with only one number in the sequence.

THREE-NUMBER BEGINNINGS

Finally, if the beginning sequence contains three or more numbers, the solution becomes quite complicated. For example, in the classical game of Nim in which the sequence contains three numbers and in each play, any one number can be reduced an arbitrary amount as long as it does not become negative, the winning combinations are obtained as follows: Let each number in the sequence be expressed as a sum of powers of 2. The sequence is a winning combination if each given power of 2 occurs an even number of times in the three representations.

The teacher's role in the classroom use of these games is to determine and maintain a maximum payoff in mathematical learning. This necessitates a balance of such factors as the capacity of the player to analyze, and his motivation to develop winning strategies. A game must be difficult enough to challenge and stimulate thought and analysis yet simple enough to make solutions possible. Thus the teacher decides when to suggest a new variation, a different set of conditions, to maintain motivation and interest and to provide the opportunity for the child to generalize and apply a strategic principle to a new game. He must also decide how best to obtain the dual values of independent discovery of strategy and the systematization of the conjectures and verification necessitated by a convincing argument as to why it works.

POLYOMINOES

(Prepared by R. P. Dilworth and S. Hill)

BACKGROUND MATERIAL FOR THE TEACHER

When geometry is applied in the physical sciences the relevant feature of a physical object is its shape. If an object is moved to a different position in space, its geometric properties are not altered. Hence, in the study of geometry, it is important to know whether or not a given configuration can be made to match another configuration by means of such a motion. It may even happen that a given configuration can be made to match itself under certain motions. In this case, the configuration will have certain symmetries. These symmetries, in turn, may determine important physical properties of objects having this shape.

In this unit a class of simple configurations will be introduced and the ways in which these configurations can be matched and fit together will be considered. These activities will provide a natural introduction to some of the basic ideas of geometry.

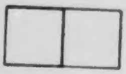
The configurations to be studied are those configurations in the plane which can be constructed by joining congruent squares along edges. Such configurations are called polyominoes. If n squares are used in constructing the configuration, it will be convenient to refer to the polyomino as an n-square.

The simplest configuration consists of a single square, i.e., the monomino or one-square. With regard to the motions of these configurations, there are two observations which should be made.

1. If two congruent squares are given in a plane, it is always possible to take one of them, turn it, and then move it across the plane in such a way that it matches exactly the second square.
2. If a square is given a quarter turn (clockwise) about its center, then it exactly matches the square in its first position. Likewise, a half turn, three quarter turn, or full turn about the center also brings the square into a matching position.

It may also be noted that a one quarter turn counterclockwise produces the same result as a three quarter turn clockwise.

The next simplest configuration is the domino or two-square which is obtained by joining the edges of two congruent squares.



The matching of dominoes is quite similar to the matching of single squares.

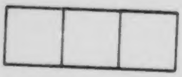
1. Given two dominoes, it is always possible to take one of them, turn it, and then move it across the plane in such a way that it matches the second domino.

A domino no longer matches itself if it is turned a quarter turn about its center. The appropriate observation for dominoes is the following:

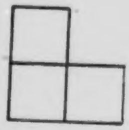
2. If a domino is given a half turn or full turn about its center, it exactly matches the domino in its first position.

The plane configuration constructed by joining the edges of three squares is called a tromino or three-square.

One such configuration is the following:

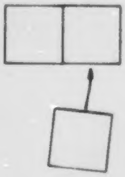


There is still another possibility

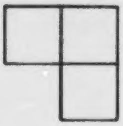


It is clearly impossible to turn the first tromino and move it across the plane in such a way that it matches the second tromino. Thus, unlike dominoes, it is not true that every pair of trominoes can be matched by suitably turning and moving one of them. On the other hand, this suggests the question: Can every other tromino be suitably moved so that it matches one of the two trominoes shown above?

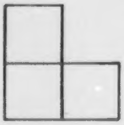
An answer to this question requires an examination of all of the possible three-square configurations. Note, first, that any tromino can be constructed by adjoining a square to a domino so that the edge of the square matches an edge of a square in the domino. If the matching edge lies at the end of the domino we get a tromino which matches the first tromino shown on previous page. If the matching edge lies along the side of the domino, then by turning and moving it across the plane it can be matched with the second tromino.



makes



and a
half turn
gives

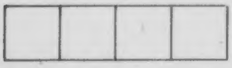


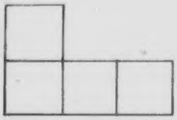
Thus there are exactly two distinct types of trominoes under motion in the plane, the straight tromino and the right tromino.

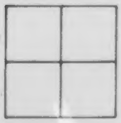
Like the domino, the straight tromino matches itself if it is given a half turn or full turn about its center. This is not true of the right tromino. It only matches itself after a full turn.

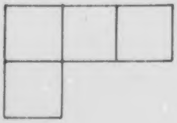
The next stage in the study is, clearly, the investigation of four-square configurations. They are called tetrominoes or four-squares. As in the previous case, the aim will be to find configurations which are distinct under the motions in the plane and to describe the motions which will take a given configuration into itself.

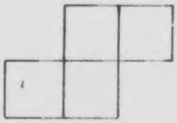
As before, all tetrominoes can be constructed by adjoining a square to a tromino. When all of the possibilities are investigated it turns out that there are seven distinct types.

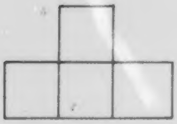


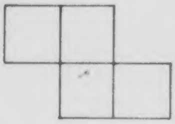








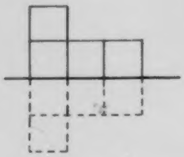


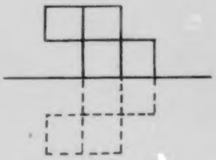


Every four-square configuration by a suitable motion in the plane can be made to match one of these seven.

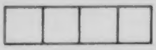
The first configuration matches itself under a half turn or full turn. This is also true for the sixth and seventh configurations. The second configuration matches itself if it is given a quarter turn, half turn, three quarter turn, or full turn. The other three configurations only match themselves after a full turn.

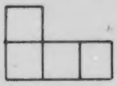
The seven distinct tetrominoes, of course, cannot be matched by turnings and motions in the plane. Nevertheless, they do not appear to be as dissimilar as the two distinct trominoes. For example, the fourth and fifth tetromino shown above are quite similar. Likewise, the sixth and seventh are alike in many respects. In fact, let us suppose that the fourth tetromino were cut from a piece of cardboard and placed on a table. Then if it were flipped over, it could then be moved on the table so that it would match the fifth tetromino. In a similar manner, if the sixth tetromino is flipped over, then it can be made to match the seventh tetromino.

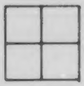


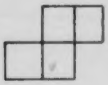


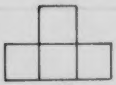
It is thus clear that, if in addition to the motions in the plane, the configurations are also permitted to be flipped over in space, then the number of distinct four-square configurations is reduced to five.











This illustrates a very basic principle of geometry; namely, the geometric relations of configurations depend upon the motions which the configurations are permitted to assume. Thus different sets of motions will normally lead to different geometric properties.

It should also be observed that if flips in space are permitted, then there may be some more motions which will make a given configuration match itself. For example, if the square configuration (the second tetromino) is flipped about either of the lines through its center, then it matches itself. Likewise, if the third configuration is flipped about a vertical line down its center, it likewise matches itself.

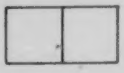
With the construction of the tetrominoes, all of the essential features of the construction of polyominoes of higher order have been encountered. Nevertheless, it is worthwhile to include some study of pentominoes and hexominoes, since these configurations exhibit the features in a slightly more complicated situation. A pupil who can recognize quickly and easily when two pentominoes can be made to match in the plane or in space has developed a good sense of configuration relations in the plane and in space. Since there are 18 distinct pentominoes in the plane and 12 distinct pentominoes in space, it is not easy to obtain all of the configurations by a random or haphazard procedure. Hence, there is a distinct advantage to the pupil who recognizes that he should plan a systematic approach to the problem. This is an important aspect of mathematical activity. Finally, the construction of hexominoes from two trominoes or from a domino and a tetromino leads to questions whose answers require both an understanding of configuration relationships and some elementary, but basic, mathematical reasoning.

TEACHING UNIT

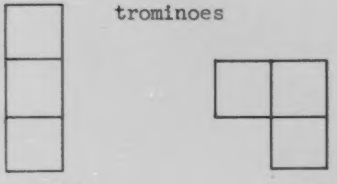
ACTIVITY I: Construction of two-square (domino), three-square (tromino) and four-square (tetromino)

Materials: Eight $\frac{3}{4}$" cardboard squares and tracing paper for each child, and/or: Demonstration for the class may be done using an overhead projector or a board such as flannel board or magnetic board. Use individual unit squares to construct the polyominoes. You may also wish to prepare cutouts or transparencies of the following polyomino configurations, (two of each are recommended), in which unit squares match the individual squares in size.

domino



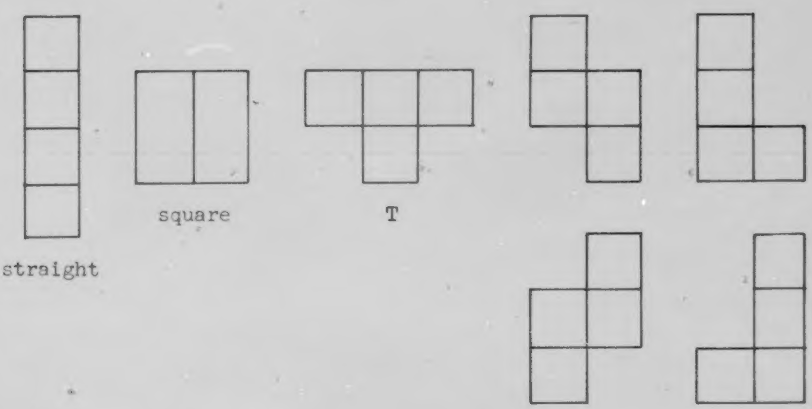
trominoes



straight

right

tetrominoes



straight

square

T

skew

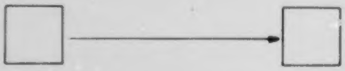
L

Monomino

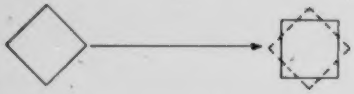
Introduce the monomino (one-square) by asking children to match their squares by fitting one upon another. Observations such as the following may be made.

1. All edges fit for any two of the squares.
2. Given two squares which fit, it is possible to turn one so that it still fits the second.
   (demonstrate a quarter turn, a half turn and a full turn)
3. Given two different squares, we can see if they match by sliding one across the board or table and turning it to see if they fit exactly. (demonstrate)

   (a) a slide



   (b) a slide and a turn



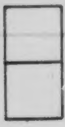
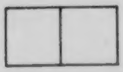
Domino

Question: Can you make a two-square by joining two of the squares so that the edge of one fits exactly the edge of another?

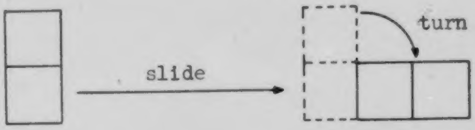
The children form the two-squares in as many ways as possible, observing that any edge of one square fits any edge of the other.

Question: Two-squares are the same shape if they can be made to fit one another exactly. How many different shapes can two-squares have?

As two-squares are formed, shapes may be traced and the different examples matched. The children observe that any of the two-squares they construct can be made to fit any other by a combination of turns and slides in the plane. In short, there is only one shape or form for the two-squares. The children should verify this by experimentation. For example, suppose a child suggests that there are two distinct two-squares

 and 

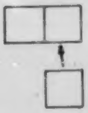
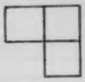
His suggestion may be tested by moving one domino cutout to try to fit the other or by tracing one pattern (if he has put together two individual squares) and moving the tracing to fit the other. Thus by a slide and a turn he demonstrates that the form is the same (they match exactly) and concludes that they are not distinct shapes.

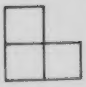
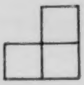


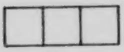
Trominoes

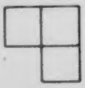
> **Puzzle:** Three-squares are made from three of the squares, by joining the edge of any one to the edge of another so that the edges match exactly. If two three-squares can be made to fit one another by turning and sliding they are the same shape. How many different shapes can a three-square have?

The children experiment by trying different combinations either individually at their own tables or by demonstrating solutions at the board or with the overhead projector. They observe that they must begin by forming a two-square and that there are several possible places to add the third square. (In fact, there are six.) Encourage them to try various possibilities and compare them by the "fitting" test to eliminate those which can be made to fit another.

As an example,  makes  which is seen to match

 by a half turn, or  by a quarter turn, etc.

But no combination of slides and turns can make 

match 

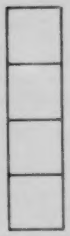
The children continue experimenting and testing the possibilities until they are convinced that there are exactly two distinct shapes a three-square can take. One is called a "straight" three-square, the other a "right" three-square.

Tetrominoes

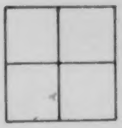
Puzzle: Four-squares are made from four of the squares, by joining the edge of one with the edge of another so that the two edges match exactly. If two four-squares can be made to fit one another by sliding and turning then they are the same shape. How many different shapes can a four-square have?

The children experiment with the possibilities, testing by slides and turns, until they find all the distinct four-squares. When we permit only slides and turns to fit two four-squares together we find seven distinct shapes.

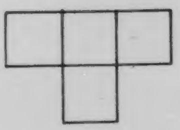
(a)



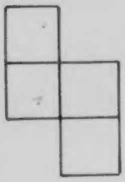
(b)



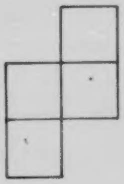
(c)



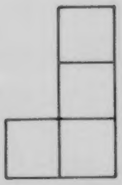
(d)



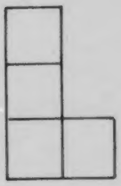
(e)



(f)



(g)



As with the three-squares, encourage the children to continue trying possibilities until they are reasonably convinced that all the distinct four-squares have indeed been found. To be convinced, of course, they need to be convinced that they have considered all possibilities. This requires a systematic, not a haphazard approach. For example, it is obvious that a four-square can always be formed by adding a square to a three-square. Since we know that there are just two distinct three-squares, we can consider all the places a square can be joined to each. Beginning with the straight tromino, the addition of a square gives us (a), (c), and (g) above. The addition of a square to the right tromino produces (b), (d), (e), and could also produce (c), (f), and (g). It is not suggested that a systematic approach be pressed upon the children. Some will proceed systematically without urging. However, some may, with a little guidance, be encouraged to try to organize their approach to the problem rather than proceed at random. In general, they should be free to take whatever approach seems appropriate to them.

> Note: Some children may notice that certain pairs of the four-squares look very much alike, (d) and (e), and (f) and (g). They cannot be made to fit by turning. It is possible that a child may suggest that he can make them fit by flipping one over. This is true, of course, and the "flip" motion is considered in the next activity. If a child discovers this, continue on to the question of how many distinct four-squares can be found when we use slides, turns and flips. The answer, to be developed in Activity II, is that whereas we find seven distinct four-squares using slides and turns, if we also permit flips, there are just five distinct four-squares.

Puzzle: Which of the four-squares can be made from two two-squares?

Use two-square cutouts or tracings for the children's demonstrations of the answers. The two squares in each should be joined so that they may not be separated. Those that can be constructed from two dominoes are (a), (b), (d), (e), (f), and (g). Notice that only one type, (c) cannot.

SPECIAL PUZZLES AND GAMES USING 3 × 3 SQUARES

Show the children a 3 × 3 square or 3 × 3 board.

1. Can you form a 3 × 3 square using only three-squares?

This puzzle will require additional materials unless you wish to solve it with the entire class by drawing a 3 × 3 square on the blackboard and letting children indicate the spaces that can be filled by three-squares by shading, tracing around, etc. If it is individual work, the child will need a 3 × 3 board (these will be duplicated on paper), three straight three-square cutouts and three right three-square cutouts.

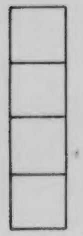
2. Can you form a 3 × 3 square using only the right three-square?

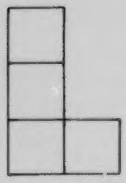
3. Can you form a 3 × 3 square using both right three-squares and straight three-squares? Can it be done with two straight three-squares and one right three-square? With two right three-squares and one straight three-square?

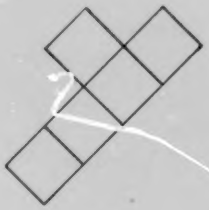
4. Game: Play on a 3 × 3 board using just two-squares. Two players take turns placing two-squares on the board. The player who places the last two-square wins. (The two-square must be placed as a unit. It cannot be separated into two distinct squares).

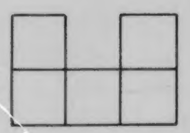
ACTIVITY II: Tetrominoes and Pentominoes under Slides, Turns and Flips

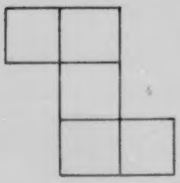
Materials: Five $\frac{3''}{4}$ squares of cardboard and tracing paper. In addition to the set of cutouts or transparencies used in the previous activity for class demonstration, you will need a set of pentominoes (five squares).

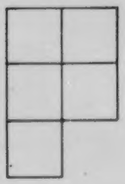

I

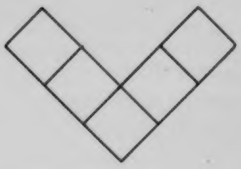

L

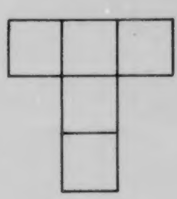

Y

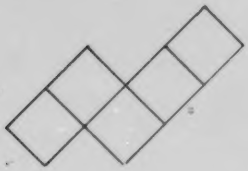

U

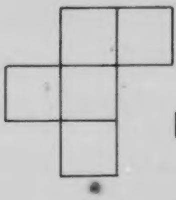

Z

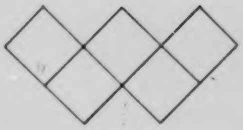

P

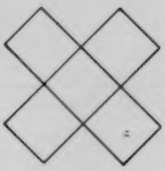

V


T


N

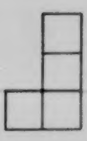
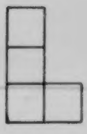

F

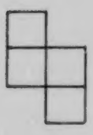
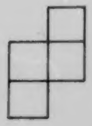

W


X

Tetrominoes and Flips

Review the seven distinct four-squares formed using slides and turns. Ask if any pair looks similar. Perhaps the children will notice that

 looks very much like 

and that  looks very much like 

Reconfirm the conclusion that no combination of slides and/or turns makes one match the other. Children can try these moves to demonstrate the impossibility of fitting one to another.

Puzzle: Can you move a four-square in any way so that it fits another?

The children experiment until they discover that if they flip the figures over, the first fits the second and the third fits the fourth. We now have three different motions which can be used in matching figures, (1) slides, (2) turns and (3) flips.

Puzzle: How many different shapes does a four-square have if we use slides, turns and flips.

Since we can combine the pairs above into just two distinct four-squares by flips, this cuts the number of distinct four-squares from seven to five. The children experiment with the construction of four-squares by sliding, turning and flipping until they conclude that there are exactly five distinct shapes.

It should be clearly understood that the number of different shapes depends on the motions permitted. If the rule says we can use slides and turns only, then seven different four-squares are formed; if the rule says we can use slides, turns and flips, then five different four-squares are formed.

Puzzle: Using slides and turns, there were exactly two different three-squares (the straight three-square, and the right three-square). How many different three-squares are there using slides, turns and flips?

Since the right and straight three-squares cannot be made to fit by slides and turns, this amounts to asking, "Can you make the two match each other by a flip?" It is fairly obvious by inspection that one cannot. However, the children can suggest their answers and try to convince the others by argument or demonstration.

Pentominoes

Puzzle: A five-square is formed by joining five of the squares. Two five-squares are the same shape if they can be made to fit each other exactly. How many different shapes can a five-square have when we use slides, turns and flips?

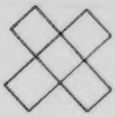
In general, the same procedure may be followed in which children, either individually at their own tables or at the board, experiment and verify. They might also work on the problem in small groups, then compare their results with the other groups.

For example, the groups can take turns suggesting a configuration by having one child construct it for the entire class. Acceptable configurations can be preserved by drawing them at the side or displaying the appropriate cutouts. The collection of distinct shapes is built up and continually visible to all. Thus when a configuration is suggested it may be challenged by another member of the class. The challenger must then demonstrate by a combination of permissable motions that it is actually the same shape as one already in the collection. This continues until all twelve of the five-squares are produced or the children are convinced that there are no other possibilities. If they have not found all twelve, suggest that you can think of others to try and give them another chance to find the others.

1. Can all the five-squares be made using a three-square and a two-square?

Again, the restriction is that neither the three-square nor the two-square may be separated into individual squares.

It turns out that only



cannot.

Game: Play on a 5 × 5 square board using just three-squares of both types. Two players take turns placing three-squares. The player who places the last three-square wins.

On a 5 × 5 board, a number of cutouts are needed. It is possible to play using a 5 × 5 square drawn on paper, where a play consists of shading in three squares. In this case, it should be clearly understood that a three-square consists of joined squares so that squares which are not adjacent to another cannot be shaded in one play.

ACTIVITY III: Symmetry

Materials: Pentomino (five-square) cutouts, paper with pentomino configurations drawn, tracing paper. (In this activity use only the L, the Z, the T, the I and the X five-squares.)

Turning Symmetries

Puzzle: How far do you turn the L five-square so that it matches itself? (the original figure)

Experimentation will show that we must turn the L five-square completely around in a full turn before it matches itself. This can be done by the child in several ways. If a drawing of the configuration is on one piece of paper, it may be traced on tracing paper and the latter turned until the traced figure again fits the original configuration. Alternatively, the configuration may be cut out of the center of a sheet of paper. Leaving the paper on the table, the figure itself is turned until it once again fits into its original slot or hole.

Using a chalkboard and a stiff cutout of the L five-square, one can trace around the cutout in one position, then turn the cutout until it matches the original tracing.

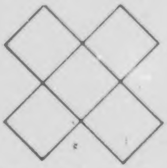
Puzzle: How far do you turn the Z five-square so that it matches itself? (the original figure)

In this case, experimentation will show that if we turn the figure halfway around, it fits its original tracing or fits into its original slot. Furthermore, a full turn also matches the original figure. The children should discover that any figure can be made to match itself under a full turn. Thus either a half turn or a full turn takes the Z-square into a matching figure?

At this point the children may note that a half turn followed by another half turn is the same as a full turn.

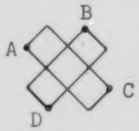
Puzzle: What turns take each of these five-squares into matching figures? the T? the I? the X?

For the T five-square, only a full turn makes it match the original; for the I five-square, both a half turn and a full turn take it into the original figure. In turning the X five-square, the children will notice that they can turn the figure just a quarter turn and it fits into the same slot. Further, a half turn, a three-quarter turn or a full turn takes it into a matching figure. Thus the X five-square



has four turning symmetries, a quarter turn, a half turn, a three-quarter turn and a full turn.

Puzzle: Some of the points on the X five-square are marked by letters.



A quarter turn makes the X five-square match itself. If you turn it a quarter turn clockwise, which point matches point A? point B? point C? point D? If you turn it a half turn, what happens to each point?

Again the children use cutouts or tracing paper to experiment. For example, if the child cuts a figure from a piece of paper on which it is drawn, he should label each point both outside and inside the cutout, as shown in Figure 1 below. Thus, when he turns the cutout a quarter turn he has both the original point A shown and the matching of A to B under the quarter turn, as shown in Figure 2 below.

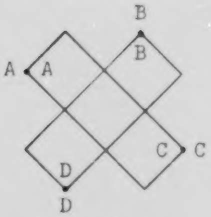


Figure 1

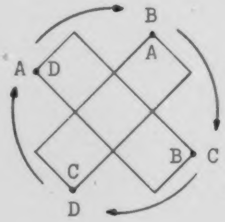
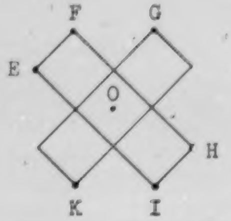
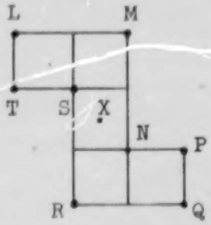


Figure 2

Under a quarter turn clockwise, A is matched with B, B is matched with C, C is matched with D, and D is matched with A.

Puzzles: What happens to each of the points marked if you turn the figure a half turn? a full turn?





Let the children try to visualize the results of a half turn, then a full turn, without actually turning the figures. They can record their guesses in some way, such as

E → H G → K I → F H → E O → O

L → Q R → M S → N T → P X → X

for half turns. For full turns, every point matches itself: E → E, L → L, G → G, etc.

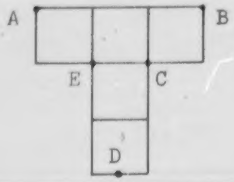
Then they can verify or correct their guesses by turning the figures. You can also devise similar puzzles for the X five-square with a quarter turn, or for the I five-square with a half turn. (You may wish to begin with simpler puzzles in which fewer points of the figures are designated.)

Folding Symmetrics

Puzzle: Which of these five-squares can be matched with themselves if they are flipped over? the L? the Z? the T? the I? the X?

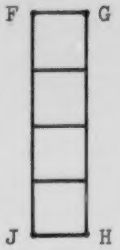
Using cutouts and/or tracing paper, the children can try flips to discover that we can flip the T, the I and the X to match the original figures, but not the L and the Z.

Puzzle: Which points match when the T five-square is flipped?



A → B, B → A, E → C, C → E, D → D

Puzzle: Which points match when the I five-square is flipped?



First record guesses, then verify by flipping the figure. Here the children discover that there are two ways to flip the figure and the matching of points depends upon which way it is flipped. For example, if it is flipped about a vertical line down the center,

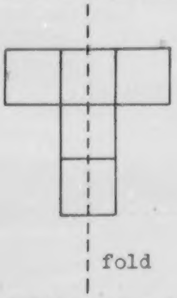
$$G \rightarrow F, \quad F \rightarrow G, \quad J \rightarrow H, \quad H \rightarrow J.$$

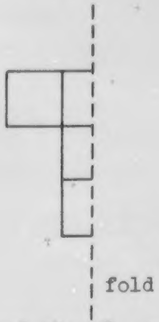
But, if it is flipped about a horizontal line across the center,

$$F \rightarrow J, \quad J \rightarrow F, \quad G \rightarrow H, \quad H \rightarrow G.$$

Puzzle: Which of the five-squares can be folded so that the lines on one side fit exactly the lines on the other side?

Probably figures drawn on tracing paper provide the best method of experimentation here. Demonstrate with a T five-square drawn on thin paper. If you fold down the center, the children see that the lines of one-half fit exactly the lines of the other.

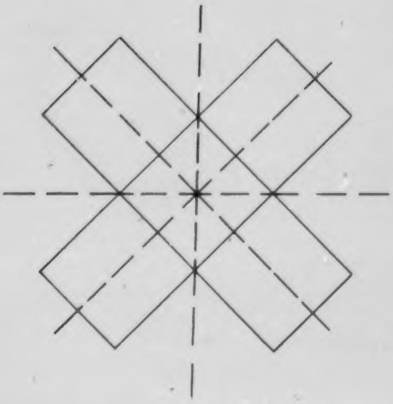




Experimentation shows that the T, the I and the X can be folded so that one-half fits the other, but not the L or the Z. If the children compare the five-squares that could be flipped to match, they discover that those which have a folding line are exactly those which could be flipped to match. Thus a figure has a line of symmetry if and only if it can be flipped and moved into a matching configuration. The children can also discover that an I five-square has two folding lines.

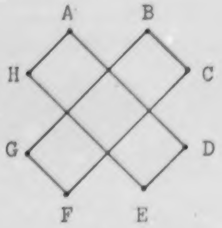
Puzzle: How many folding lines has the X five-square?

Experimentation with folding shows that there are four lines of symmetry as shown below.



SPECIAL PUZZLES: SYMMETRY OF PENTOMINOES

1. Name all points which can be matched with point A by flips.



Since there are four lines of symmetry, each flip (or fold) matches A with a different point.

A → B, or A → F, or A → H, or A → D

2. Tell whether the points in the figure above are matched by a turn or a flip if

A → C? D → E? E → F? H → D? A → A?

3. If B is matched with D (B → D), what point is matched with G?

(G → A by a quarter turn)

4. If H is matched with E (H → E), what point is matched with B?

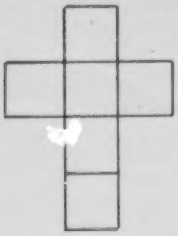
(B → C by a flip)

5. How many of the points named in the figure above can be matched with A, using either flips or turns?

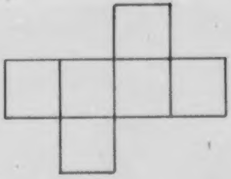
(all of the points - B, F, H, D by flips
C, E, G, A by turns)

ACTIVITY IV: Some Hexominoes

Puzzle: Six-squares are made by joining six of the squares. Two examples are:



and



Which can be made from a four-square and a two-square?

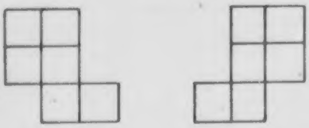
The first one, not the second one.

Puzzle: Which of the above six-squares can be made from two three-squares?

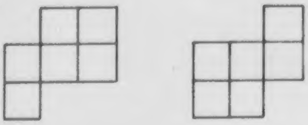
The second, not the first.

Puzzle: Which of these pairs of six-squares are different shapes and which are the same shape? If they are the same, how can you move one to fit the other?

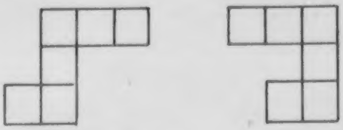
1.



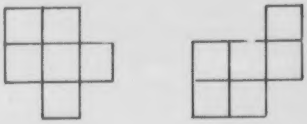
2.



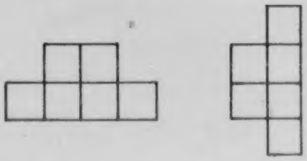
3.



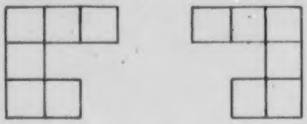
4.



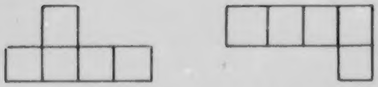
5.



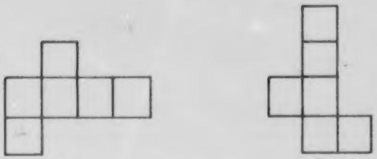
6.



7.



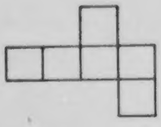
8.



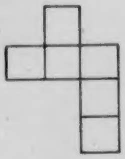
(1) same, under a slide and flip
(2) same, under a slide and turn
(3) not the same
(4) not the same
(5) same, under a slide and turn
(6) same, under a slide and flip
(7) not the same
(8) same, under a slide and turn

1. Which of these six-squares can be made from a four-square and a two-square but not from two three-squares?

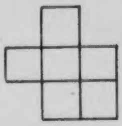
(a)



(b)



(c)



2. Use a 4 × 3 board. Can you completely cover the board using

(a) a five-square, a six-square and a one-square?

(b) using a three-square, a five-square and a four-square?

(c) using a six-square, a four-square, and a two-square?

(d) using a six-square, a five-square, and a two-square?

(e) using a five-square, a four-square, and a one-square?

The children may select any of the configurations listed; for example, to answer the first question they try any of the five-squares with any six-square and a one-square. They try fitting them together in the manner of a jig-saw puzzle until they find a combination that exactly fits the 4 × 3 rectangle.

This may be played as a game by two children who take turns selecting. The first child to select the three required polyominoes that will fit the 4 × 3 rectangle wins.

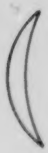
In experimenting with combinations, the child may become aware that there are 12 squares to cover. Thus it will be obvious that only a combination with a sum of 12 units could possibly fit. If he notices this, he may see immediately that the answer to (d) is "no" because the total number of squares is 13. Otherwise he may try a few combinations and finally conclude that nothing fits because there are too many squares. In either case, he may be able to apply these conclusions to question (e) and see that the answer must again be "no" because a five-square, a four-square and a one-square give a total of only 10 squares.

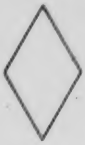
SYMMETRY

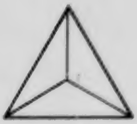
(Prepared by F. Sinden)

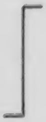
BACKGROUND ON SYMMETRY

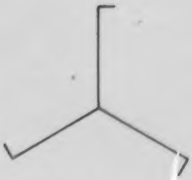
These patterns are symmetric:


1

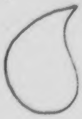

2

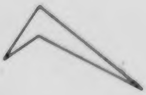

3

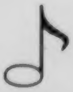

4


5

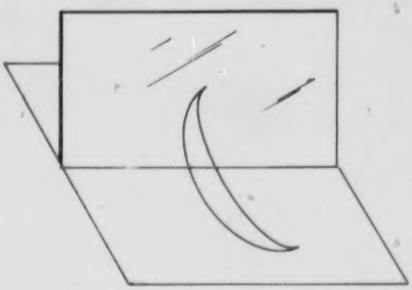
These patterns are not symmetric:





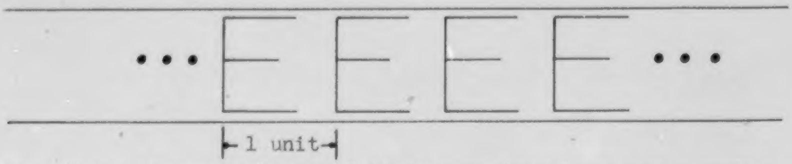


Symmetric patterns are those which look the same after being rotated, reflected or translated in some way. Patterns 2 and 4, for instance, look the same when rotated one-half turn. (To see this turn the page upside down.) Patterns 3 and 5 look the same when rotated one-third or two-thirds of a turn. Thus patterns 2, 3, 4, and 5 have rotational symmetry. Pattern 1, the lune, does not have rotational symmetry. (When rotated one-half turn, for instance, the lune looks different: the points are to the left instead of to the right.) The lune has reflective symmetry. If you place a mirror perpendicular to the paper with one edge along a horizontal line through the center of the lune as shown below:



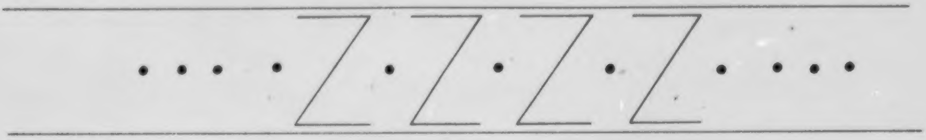
then the visible half of the lune together with its reflection looks identical to the complete lune. Thus reflective is a reasonable word to describe the symmetry of the lune. The horizontal line is the lune's line of symmetry. The two halves of any figure folded along a line of symmetry match exactly. Pattern 2 has two lines of symmetry (vertical and horizontal lines through the center) and pattern 3 has three lines of symmetry. A square has four lines of symmetry. Patterns 4 and 5 have no lines of symmetry (though, of course, they do have centers of rotational symmetry).

The definition of symmetry given above contains three key words: rotated, reflected and translated. We have discussed the first two. A figure is said to be translated if it is shifted in some direction without changing its orientation. Only infinite patterns can look the same after a translation. Imagine, for example, that this pattern is extended infinitely in both directions:



Since this pattern does not change if it is translated any whole number of units to the right or left, it has translational symmetry. (It also has reflective symmetry with respect to its horizontal center line.)

A second example is shown below:



In addition to its translational symmetry, this pattern also has rotational symmetry with respect to each of the dots. (To see this, invert the page.)

Wallpaper patterns are examples of more complex translationally symmetric patterns.

The types of symmetry are highly interrelated. A systematic study of these interrelationships leads to Group Theory, a beautiful and fundamental branch of mathematics whose implications reach far beyond the study of symmetry.

Without being systematic, here are a few of the interrelationships among symmetry types. Referring to patterns 1, ..., 5 displayed at the beginning of this discussion, note that the only patterns with more than one line of symmetry (Patterns 2 and 3) also have centers of rotational symmetry. This is no accident: *the intersection of two lines of symmetry is always a center of rotational symmetry*. The best way to see that this is true is to try to violate it, i.e., try to invent a pattern with two intersecting lines of symmetry but no rotational symmetry.

Another principle: if a pattern has two lines of symmetry and no more, then these two lines must be perpendicular. Again the way to see this is to try to violate it. (This is set as an exercise for the pupil. (Work Sheet 1 in the attached Symmetry Unit).) More generally: if a pattern has exactly n lines of symmetry with a common intersection, then those n lines must be spaced uniformly in angle about their point of intersection. A pattern which has two *parallel* lines of symmetry is necessarily infinite and has translational symmetry. This principle is used to introduce translational symmetry in the attached Symmetry Unit. (Work Sheet 2)

Translation in two directions leads to doubly infinite patterns (wallpaper). Here the possibilities multiply. It is shown by means of Group Theory that there are exactly seventeen distinct types of symmetry in doubly infinite patterns. The Moors, in the course of their artistic experimentation, discovered all of these. In fact, all seventeen are represented in the decorations of the Alhambra.

Extended to three dimensions the systematic study of symmetry is important in crystallography. This is not, however, the reason for introducing the subject to grade school students. The reason is that symmetry and abstract ideas derived from symmetry are powerful tools in understanding mathematics. In fact, symmetry comes as close to being the secret of the trade as anything. This is very visible (the secret is not well kept) in high school geometry where many of the simplest and cleverest proofs are based on some construction which renders the figure symmetric. The symmetry of the figure then renders the sought relationship immediately evident. In a more abstract way this occurs constantly throughout mathematics.

There is no reason why students should not begin to develop their sense of symmetry long before they reach high school.

References

1. For ingenious art based on symmetry see: M. C. Escher, "The Graphic Work of M. C. Escher," Meredith Press, New York, 1967.
2. For mathematical theory with many pictures see: L. Fejes Tóth, "Regular Figures," Pergamon Press, New York, 1964.

SYMMETRY UNIT

(1) Definition by Example

Display a collection of symmetrical shapes (labeled "symmetrical") and a collection of unsymmetrical shapes (labeled "unsymmetrical"). (Suggested shapes are attached.) After the students have studied the two collections for a few minutes produce new shapes one at a time to be identified as symmetrical or unsymmetrical. Do not try to define symmetry in words. Include many kinds of symmetry (stars, lunes, swastikas, etc.). P. 1, 2.

(2) Meaning of Symmetry (transformations)

Display a symmetrical shape. Stand in front of it and pretend to manipulate it. Stand back and ask, "Have I changed its position." Since the shape is symmetrical, it is not possible to tell. This serves as a definition: If a shape can be turned in some way without changing its appearance, it is symmetrical.

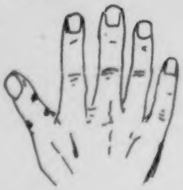
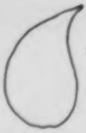
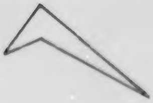
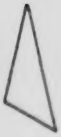
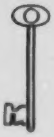
Draw outlines of some of the shapes on the blackboard, including at least one unsymmetrical one and one with only rotational symmetry. A symmetrical shape can be fitted into its outline in more than one way, but an unsymmetrical one cannot.

(3) Lines of Symmetry

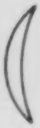
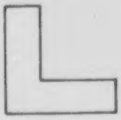
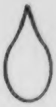
Observe that some symmetric shapes can be folded along a line so that the two halves match. Demonstrate with a few of the cutouts. True of all symmetric shapes? Ask students to try to find a line of symmetry (fold line) for some shapes with only rotational symmetry (parallelogram, swastika).

SOME SUGGESTED SHAPES

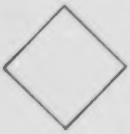
No symmetry:

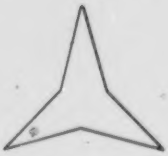
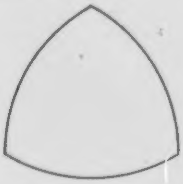
One line of symmetry:

Two lines of symmetry:

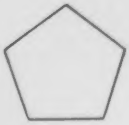


Three lines of symmetry:

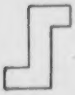
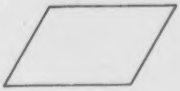
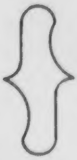
Four lines of symmetry:

Five lines of symmetry:

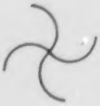


∞ lines of symmetry:

2-Fold rotation:

4-Fold rotation:



(Mirror mounted on block.) Observe that if a mirror is placed along a line of symmetry, the visible half of the figure together with its reflection looks the same as the figure itself.

Observe that shapes with a line of symmetry fit into their outlines when flipped over. (Shapes with rotational symmetry fit into their outlines when turned like a pinwheel.) You have now tested for lines of symmetry three ways: (1) by folding, (2) by reflecting in a mirror and (3) by flipping. Establish in discussion that the three tests are equivalent. A good way to do this is to challenge the pupils to find a shape that passes one test (e.g., folding) but fails another (e.g., reflecting). Of course, this is impossible. Do some shapes have more than one line of symmetry?

(4) Work Sheet 1 (attached)

Pass out work sheet 1 for pupils to do in class. Go around class with mirror checking worksheet 1. Demonstrate errors by placing mirror along erroneous lines of symmetry.

(5) Homework

Find (capital, block) letters of the alphabet with,

(a) no symmetry.
(b) one line of symmetry.
(c) two lines of symmetry.
(d) rotational symmetry but no line of symmetry.

(6) Game

Equipment: 8 × 6 grid of squares divided by a heavy line into two 4 × 6 halves, 12 pasteboard squares each of such size as to cover exactly four squares of the grid.

Rules: Players take turns placing squares on grid. Each square must cover exactly four grid squares. Once a square is put down it is not moved.

No square may overlap the heavy center line, and no square may overlap another square which has been put down before, and no square may extend outside the grid.

A player wins when the other player cannot find a place to put his square.

Strategy: The second player can always win by placing his square symmetrically to the last square placed by the first player. The simplest symmetry to use is reflective symmetry in the heavy center line, but one can also use rotational symmetry about the center or a translation.

Explanation: Just after the second player's move the pattern of squares is always symmetric, and therefore so is the pattern of empty spaces. If there is space for a square on one side of the center line, there must also be a space for a square on the other side. If player 1 can play once more, so can player 2.

Play: Using a flannelboard play the game with individual students before the class with the student going first until the students catch on to the winning strategy. Get them to explain how the strategy works. This is a good exercise in expressing mathematical ideas.

(7) Work Sheet 2 (attached)

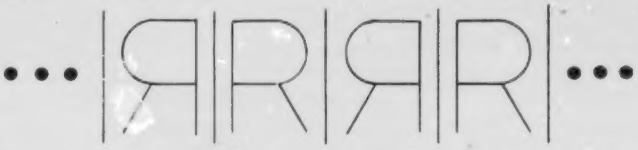
Use mirror to check as with work sheet 1.

(8) Rotational Symmetry

Pick out all the shapes with rotational symmetry but no line of symmetry. Observe that they fit into their outlines when rotated like a pinwheel. Discuss shapes with 2-, 3-, 4- ... fold rotational symmetry. Now find rotationally symmetric shapes that also have lines of symmetry. Count the lines of symmetry in each case. Observe that there is always more than one line of symmetry. Can a rotationally symmetric shape have one and only one line of symmetry? (No.) Challenge the pupils to draw a shape with 2-fold rotational symmetry and one and only one line of symmetry. When they are convinced it is impossible, ask for an explanation.

(9) Translational Symmetry

Draw two vertical parallel lines on the board. Draw an "R" between them. Can you complete this figure so it is symmetric in both lines? Leads to infinite pattern:



How far do you have to slide the pattern to make it look the same? Find a translationally symmetric pattern with no lines of symmetry (make all the R's face the same way). Find one with a horizontal line of symmetry (use E's), rotational symmetry (use Z's), etc.

(10) Symmetric Patterns in the Infinite Plane

Get some wallpaper samples (preferably with a small pattern) large enough to cover a couple of cycles both ways. Find and mark centers of rotational symmetry, and lines of (reflective) symmetry. Note which wallpaper samples give the same arrangement of lines and centers.

(11) Homework for Fast Pupils

(a) Find a figure that is symmetric with respect to both of two perpendicular lines.

(b) ... and not with respect to any other line.

(c) Find a figure that is symmetric with respect to both of two non-perpendicular lines (circle).

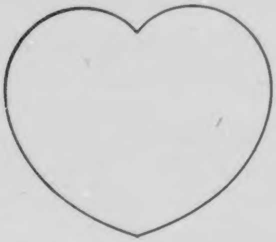
(d) ... and no other (impossible).

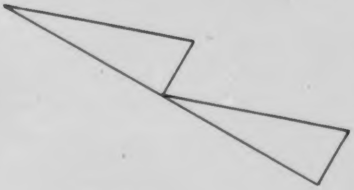
(e) Three non-concurrent lines.

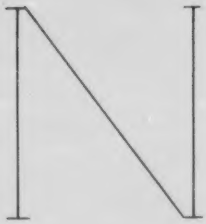
(f) What can be said about simultaneous lines of symmetry?

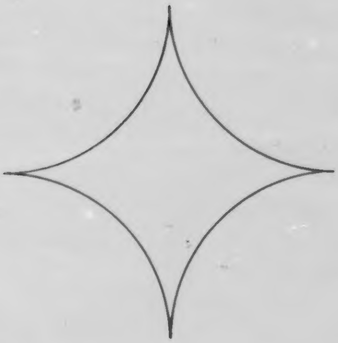
WORK SHEET 1

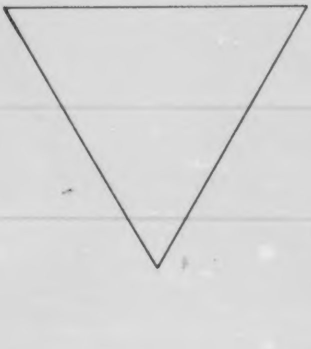
Draw all lines of symmetry.

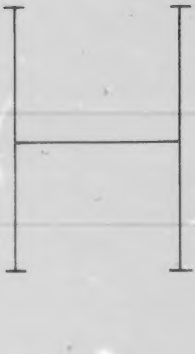






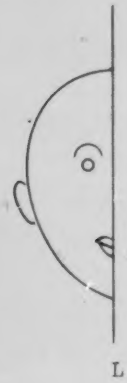


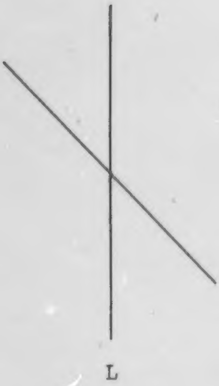


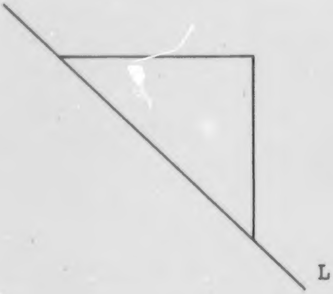


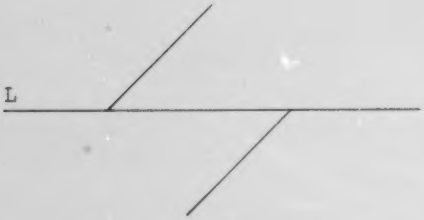
WORK SHEET 2
(revised)

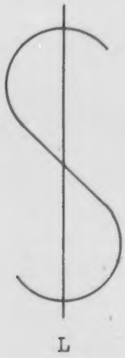
Complete each pattern so that it is symmetric in line L. Be neat and accurate.

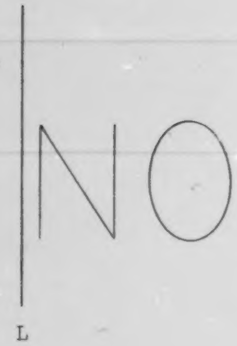


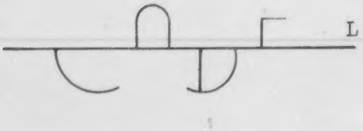












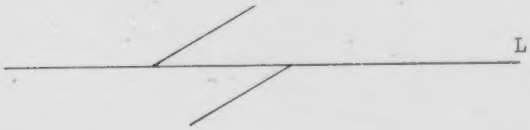
Report on Trial in Summit, N. J.

The class was a fifth grade, said to be slower than average. The trial consisted of one 30 minute and one 40 minute session a week apart. Items 5, 8, 9, 10 and 11 and the folding in item 3 were not included.

None of the students knew the word symmetry at the outset. Most did not get its meaning until it was pointed out (item 2) that symmetric shapes look the same in two (or more) positions. Many could see without the mirror which shapes had lines of symmetry but many others could not and seemed surprised to find that they could not place the mirror on a swastika so as to preserve its shape. Because of this difficulty I have included the folding in item 3. The mirror, though, is valuable because it leaves the figures flat and shows up errors dramatically. It also leads naturally to the standard terms, "reflection" and "reflective symmetry."

Work Sheet 1: All students found the line of symmetry for the heart without difficulty. Many put a vertical line on the N and were surprised to see the mirror transform the N into an M. Most found the vertical and horizontal lines for the 4-pointed star but failed to find the diagonal lines. Similarly, most found only one of the 3 lines of symmetry for the equilateral triangle. A few students got the whole work sheet right without help. This was a successful exercise but it should be used in class with coaching.

Work Sheet 2: The first two gave no trouble. Most were puzzled by the pair of V's separated by a horizontal line, some even when aided by the mirror. This has now been replaced by:



Most made the S into an 8 but not very exactly. I'm not sure how much exactness to demand. About half got the word CODE without prompting. This one was harder than I expected.

Game: The game was played on a Vugraph, but a flannelboard would be better. It generated lots of excitement and noise. I started out playing the less obvious strategies (rotation instead of reflection) and switched strategies each time. This precaution was unnecessary. Although the students quickly began to act as if they had caught on, in fact they had not. It would be better to start right out with the (seemingly) obvious reflection symmetry and stick to it until they catch on. Then play other strategies (i.e., use other transformations). The students soon concluded that it is advantageous to go second. It takes fast thinking to win if you go first and your opponent fails to play a symmetry strategy, but it can usually be done. Your winning demonstrates effectively to the student that he has not yet found the secret.

Even after the students knew how to win, it took considerable coaxing to get them to understand clearly why the symmetry strategy works. Probably less than half understood it at the end. The reasoning goes as follows: If the pattern of squares is symmetric then so is the pattern of spaces. Therefore, if there is a space for the first player, there must also be a space (on the other side) for the second player.

Palo Alto Trials

It was obvious from the Palo Alto trials that this unit puts unusual demands on the teacher because the subject is unfamiliar. The material will be successful only if the teacher prepares herself conscientiously. In particular, she should work through all the questions and problems in detail and study the shapes thoroughly.

This was the main conclusion; following are a few specific observations.

The mirror was a very valuable prop, especially in showing up errors on the work sheets. When an N turns into an M the effect is dramatic and convincing. The light was often seen to dawn on such occasions.

The parallelogram is a good shape for distinguishing rotational from reflective symmetry. Many children and some teachers have to do quite a bit of folding and reflecting before they are convinced that it has no line of symmetry.

Outlines to fit the shapes into should be drawn beforehand. It is too distracting and time-consuming to do it during the discussion.

Asking children to do impossible things seems to be a good device. Children do not like to accept impossibility and they rack their brains to get around it. This confirms my belief in the power of negative thinking.

CLASSROOM ACTIVITIES BASED ON COMPUTER CONCEPTS

A COUNTING MACHINE

(Prepared by W. Jacobs and J. Henkelman)

INTRODUCTION

The purpose of this activity is to have students learn certain aspects of number representations in the course of carrying out a game-like classroom exercise involving a human counting machine. There are three major phases to this activity:

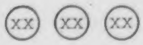
1. The teacher leads three students who act as the parts of a counting machine able to count to 7. In this phase (which should take 5-10 minutes), the emphasis is on getting the "machine" to perform correctly and in getting the class to identify the relation between the number counted and the appearance of the "machine parts."
2. When the machine is operating fairly smoothly, the teacher asks questions that lead the class to identify the number for any machine representation and the machine representation for any number (if it is possible). This phase also lasts about 5-10 minutes.
3. Finally, the teacher asks questions which the students can answer by operating the machine correctly. However, the questions can also be answered by understanding the relationship between the representations on the counting machine and the binary representation of numbers. These exploratory questions are excellent for developing the kind of thinking many educators think is very valuable.

BINARY REPRESENTATIONS OF NUMBERS

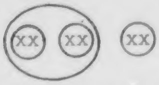
First, let's take a look at the way we represent numbers. When we write 134, we are using the 1 to represent one group of hundreds, the 3 to represent three groups of ten, and the 4 to represent four ones. We can emphasize this by writing $100 + 30 + 4$ or $(1 \times 100) + (3 \times 10) + (4 \times 1)$ or $(1 \times 10 \times 10) + (3 \times 10) + (4 \times 1)$. As we are all aware, this representation of numbers is based on ten. If we wanted to represent a number such as one thousand, one hundred, thirty four, we need to annex another 1 and write 1,134. The 1 which was annexed represents one group of thousands or 1×1000 or $1 \times 10 \times 10 \times 10$.

In this activity, the counting machine will represent numbers using a binary system based on two rather than the usual representation based on ten. Now when we write 10 in the binary representation, the 1 represents one group of two, the 0 represents zero ones. Thus, the binary numeral 10 represents the number two. In the binary representation 110, the first 1 represents one group of fours, the second 1 represents one group of twos, and the 0 represents zero ones. Hence, the binary numeral 110 represents the number six. What would the binary representation 101 be? The first 1 represents one group of fours, the 0 represents zero groups of two, the 1 represents one one. Hence, the binary 101 is simply another name for five. We can write the binary representation of 101 as (1 × 4) + (0 × 2) + (1 × 1) or (1 × 2 × 2) + (0 × 2) + (1 × 1). How would you write the binary representation 1010 as above? The binary representation 1010 is (1 × 2 × 2 × 2) + (0 × 2 × 2) + (1 × 2) + (0 × 1) or (1 × 8) + (0 × 4) + (1 × 2) + (0 × 1) or simply ten.

Let's take a look at this from the other direction. How would we represent seven as a binary numeral? Now we need to group by two's as follows:



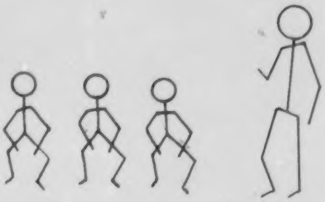
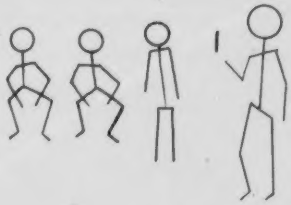
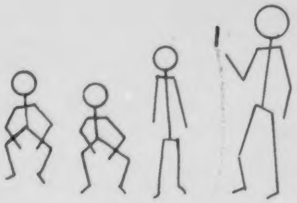
x. Now there needs to be another grouping of the groups by two's as follows:

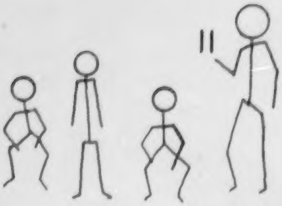
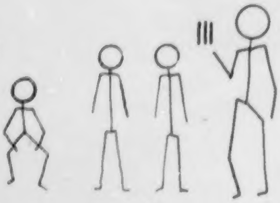


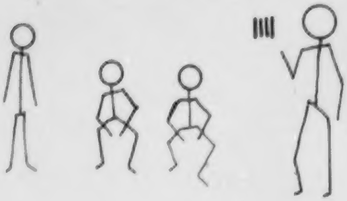
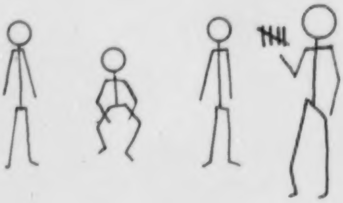
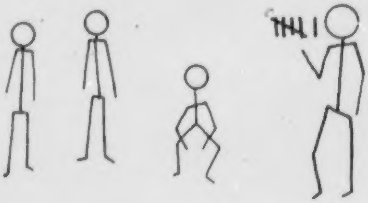
x. This could be written as (1 × 2 × 2) + (1 × 2) + (1 × 1) or (1 × 4) + (1 × 2) + (1 × 1). The binary representation for seven would be 111.

Let's get on with the activity and see how we can use students as "machine parts" of a counting machine to get a binary representation of the numbers.

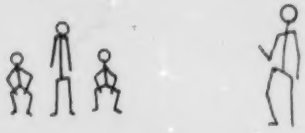
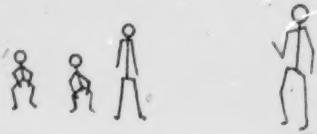
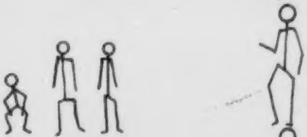
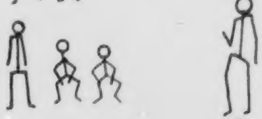
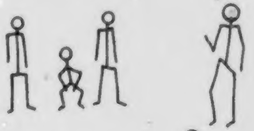
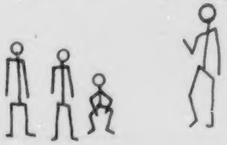
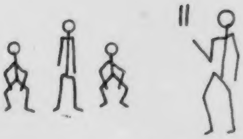
ACTIVITY: Phase 1

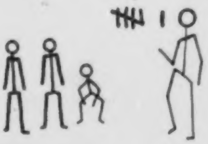
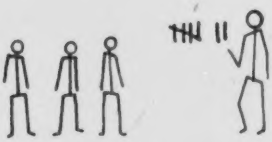
Teacher says:	Classroom appearance:	Comments:
"Has anyone ever seen a counting machine?"		Discuss students' experiences pointing out mechanical nature and parts of suggested counting machines.
"Today we are going to make our own counting machine. In fact, the moving parts of our counting machine will be some of you."		Select three students and arrange in three chairs at front of room.
"Now let's see how the parts of our machine will move when it is counting." "Now the machine will count to one."		Write a tally mark on the board. Pass the chalk to the student seated in the first chair. Instruct the student to stand and pass the chalk to the student in the second chair. Instruct each student to pass the chalk until you get it from the student in the third chair.
"This is the way the machine names the number one."		It is important to establish the relationship between each number counted as recorded in the tally and the machine representation of each number.
"Now the machine will count to two."		Before counting to two, be sure that no one has changed positions since you counted to one. Write a tally mark on the board. Pass the chalk to the student standing at the first chair. Instruct him to change positions and pass the chalk to the second student.

Teacher says:	Classroom appearance:	Comments:
		Now instruct the second student to change position and stand. The third student once again gets the chalk, remains seated, and passes the chalk to the teacher.
"This is the way the machine names the number two." "Now the machine will count to three." "This is the way the machine names the number three."		Repeat the same procedure for one more tally.
"This is the way the machine operates. I write a tally on the board with a piece of chalk. This is the number which is being counted. I pass the chalk to the student at the first chair. He always changes his position and passes the chalk to the student at the next chair. This second student and all other machine parts simply watch the student who passed them the chalk. If the student who passed the chalk sat down, the next student changes his position and passes the chalk. If the student who passed the chalk stood up or remained unchanged, the next student stays the same and passes the chalk. I get the chalk from the last student, and the machine has now finished counting the tally."		After instructing the machine for several tallies, then the rules for the motion of each part should be given. As you can see the sitting down of a student is simply a signal for the next student to change position. This sitting down is like the carrying which takes place in adding. It acts in the same way as the mileage indicator on a car; when the right hand dial moves from nine to zero, the next dial is advanced one digit.

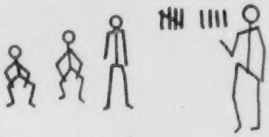
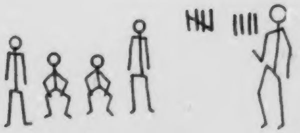
Teacher says:	Classroom appearance:	Comments:
"Let's use these rules for each machine part and count to four." "This is the way the machine names the number four."		Follow the rules just stated in counting several more tallies. You may ask the class to comment on whether a particular machine part is moving correctly. It is easier for the observers to correct the machine's operation than it is for the machine itself. This is also an excellent way to involve the entire class in the process. You may want to change the machine parts after you have succeeded in getting one machine to work properly. A second machine will ordinarily work properly much more quickly. In selecting the machine parts, it is necessary that the first student be able to keep the operation moving. Each machine part is important, but smooth counting machine operation is lost when the first machine part does not respond to the tallies.
"Now let's count to five." "This is the way the machine names the number five."		
"And now six." "This is the way the machine names the number six."		

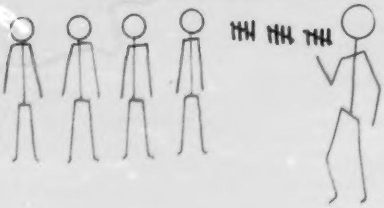
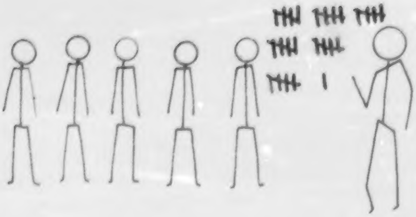
ACTIVITY: Phase 2

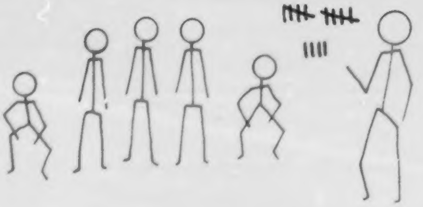
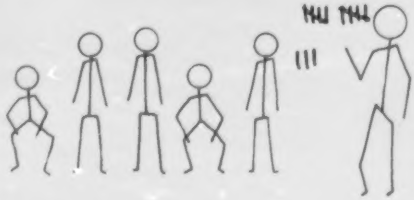
Teacher says:	Classroom appearance:	Comments:
"Now let's see if we can tell what number the machine is naming. What number is the machine naming when it looks like this?"		You now control the parts of the machine to make the representations you want. If the students in the class are not able to identify the representations either by memory or by using the base two, you can always begin the counting again from zero until you get to the representation you want. Do not attempt to explain the relationships. Give the students an opportunity to discover whatever relationships they can for themselves by your careful selection of examples. Note that the suggested order in which the representations are presented emphasizes that two is the base and that the representations of each machine part are added.
"What about this?"		
"What number is the machine naming when both these machine parts are standing?"		
"What number is this?"		
"And now this?"		
"And this?"		
"Now let's see how the machine will name a number we pick. Let's try two. Who should be standing?"		The students in the class should be able to respond to these with little difficulty.

Teacher says:	Classroom appearance:	Comments:
"Let's try naming six on the machine. Who should be standing?"		You might let the class suggest numbers to represent on the machine. If you do this, and any number greater than seven is mentioned, you will be ready to move to the Phase 3 activity.
"Who should be standing if we name seven on the machine?"		

ACTIVITY: Phase 3

Teacher says:	Classroom appearance:	Comments:
"How would the machine name the number nine?"		You are now ready to ask some of the questions which involve the students in extending their observations to new situations with the counting machine. Since the students are not likely to be able to answer these questions immediately, you should begin with a number representation they know -- such as seven -- and count to nine using the rules.
"This is not very satisfactory since this is also the way the machine named one. What could we do to the machine in order to be able to name nine in a different way than one?"		
"Let's add another machine part."		The students will ordinarily suggest that if another machine part is added to the machine the number nine can be named in a different way than one. Add another student as a machine part and begin counting from seven again.
"Now let's count from seven."		

Teacher says:	Classroom appearance:	Comments:
"What would be the largest number we could name with this machine?"		Here the students should be given a chance to guess. The correct answer is fifteen. You can handle the responses in a variety of ways. You can count and check the responses by simply following the rules for the machine, or you can examine the representations for each machine part.
		You can now return to the Phase 2 type activity and ask for representations of numbers on the machine and for the numbers represented by the machine with the additional machine part.
"What would be the largest number we could name with a machine which had five machine parts?"		Additional machine parts may be added and the many possibilities explored. It is impossible to predict all the directions which this activity will take. You should be open to the variety of responses from the students so that they will be able to freely explore the relationships involved in the binary representation of the numbers by the machine.

Teacher says:	Classroom appearance:	Comments:
"Now let's name the number fourteen on the machine."		One interesting change in the rules might be explored. If each machine part after the first one changed position only when the machine part who passes the chalk to him stands up, we have a reversal of the counting procedure. This new rule enables us to count backwards.
"Change the rules for the machine so that each part after the first one changes position when the machine part who passes the chalk to him stands up, instead of when he sits down." "What number is named by the machine now?"		

There are many directions in which this activity might lead. The counting machine will illustrate one aspect of computing devices clearly -- complex procedures can be expressed by the repeated use of simpler ones. This is important for the teaching of arithmetic. In addition, in order to get the counting machine in good working order a good deal of team work is involved. Hence, the counting machine teaches both arithmetic structure and team work. Furthermore, such activities provide a valuable preparation for the eventual study of computer programming; this is already taught to some high school students.

Third or fourth grade seems ideal for this activity. The activity has been tried with all elementary grades. Not much success was experienced in the first grade since the students have a rather short attention span. First grade students are unable to pursue the activity far enough to get to any interesting relationships.

The following minimum objectives to expect of the children at the completion of classroom activity with the counting machine are suggested:

1. The child should be able to identify and name the number represented by the counting machine.
2. The child should be able to represent a number on the counting machine.
3. The child should be able to demonstrate a procedure for counting to a number with the counting machine.

APPRAISAL

Select a sample of four or five students about whom you are unsure as to whether they have acquired the behaviors of the exercise described in the objectives.

(Obj. 1) 1. Arrange a counting machine so that it represents the number four. Ask each student individually, "What number is named by the machine?"

(Obj. 1) 2. Arrange the machine so that it represents the number eleven. Ask each student individually, "What number is named by the machine?"

(Obj. 2) 3. Make available a counting machine with four machine parts. Ask each student individually, "Arrange the machine so that the machine names the number eight."

(Obj. 2) 4. Make available a counting machine with four machine parts. Ask each student individually, "Arrange the machine so that the machine names the number ten."

(Obj. 3) 5. Make available a counting machine with four new and inexperienced machine parts. Write the numeral 6 on a sheet of paper and show it to the student. Tell each student individually, "You are to take charge of this machine. Take the machine through all of the necessary steps so that the machine names this number."

CLASSROOM ACTIVITIES BASED ON COMPUTER CONCEPTS

FINDING THE GREATEST COMMON DIVISOR

(Prepared by W. Jacobs and J. Henkelman)

After you have read the following few pages, you will be able to organize a demonstration of a computer finding the greatest common divisor of two numbers using the children in your classroom as the participants.

GREATEST COMMON DIVISOR

But before we begin, let's find the greatest common divisor of two numbers -- sometimes known as the greatest common factor of two numbers. This will not be the procedure that will be followed in the computer demonstration, but it will provide good background for us. We know that 6 is a factor of 12 in the set of whole numbers because there is a whole number 2 such that $2 \cdot 6 = 12$. 5 is not a factor of 12 because there is no whole number which when multiplied by 5 is 12.

Now we can write all of the factors of 12. List them.

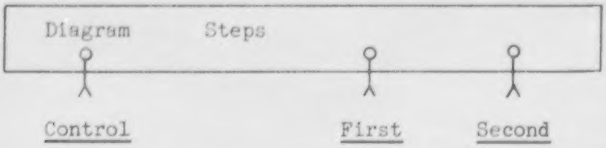
____.

Your list should include 1, 2, 3, 4, 6, and 12. These are all of the factors of 12. Can you write the set of all factors of 18? ____________ __________________. Your answer should be {1, 2, 3, 6, 9, 18}. What factors of 12 and 18 are the same? __________________. 1, 2, 3, and 6 are the factors which are the same for 12 and 18. These are called the common factors. Naturally, the greatest of these numbers is 6. 6 is known as the greatest common factor or the greatest common divisor. Let's find another greatest common divisor.

What is the greatest common divisor of 15 and 28? ________________. The answer is 1 since the set of factors of 15 is {1, 3, 5, 15} and of 28 is {1, 2, 4, 7, 14, 28}. 1 is the only common factor of 15 and 28, and therefore it is the greatest common factor or greatest common divisor of 15 and 28.

FINDING GREATEST COMMON DIVISOR WITH COMPUTER

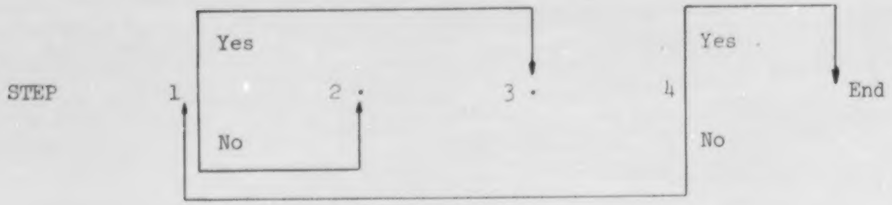
It is possible to use students as parts of a computer in order to find the greatest common divisor of any two numbers in another way. You could start by placing three students -- called Control, First, and Second -- in the front of the chalkboard as follows:



The diagram and steps already on the chalkboard are those that appear on the next page. Control directs the program which consists of only four steps. First begins by writing down one of the numbers of the pair for which the greatest common divisor is to be found; Second writes down the second of the pair of numbers. Let's find the greatest common divisor of 12 and 18 with the computer. First writes "12" on the chalkboard; Second writes "18."

In order to compute the greatest common divisor of 12 and 18, the computer must progress from step to step in the diagram following the arrows when appropriate. Control directs the program by pointing to the number of each step in the diagram as it is carried out.

DIAGRAM FOR FINDING THE GREATEST COMMON DIVISOR

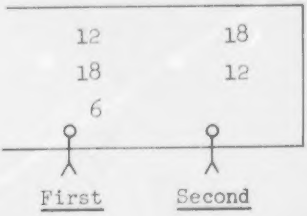


STEPS FOR FINDING THE GREATEST COMMON DIVISOR

STEP	STUDENT	ACTION
1	First	Calls "Yes" if his number is greater than or equal to Second's number, "No" if smaller.
2	First and Second	Change numbers -- First writes down Second's number and Second writes down First's number.
3	First	Subtracts Second's number from his and keeps the difference as his number.
4	First	Calls "Yes" if his number is 0, "No" if it is not 0.

Control also calls out the name of the student -- *First* or *Second* -- who is involved in action in that step. He might also read the action for that student for the step. Let's follow the activity with 12 and 18.

Control points to step 1 in the diagram and then says, "*First*," since the steps list *First* as the student who is to be active in step 1. Since *First's* number is smaller than *Second's*, *First* says, "No." *Control* then follows the arrow with his finger for the No in step 1 on the diagram and finds he is pointing to step 2 in the diagram. *Control* says, "*First* and *Second*," since *First* and *Second* are listed as both active in step 2. *First* then writes down 18, and *Second* writes down 12 in order to change the numbers as directed in step 2. *Control* then points to step 3 in the diagram and says, "*First*." *First* subtracts *Second's* number from his and writes the difference. The chalkboard in front of *First* and *Second* would look like this at this point:



Control then points to step 4 in the diagram and says, "*First*." *First* calls, "No," since his number is now 6, not 0. *Control* then follows the arrow for the No in step 4 in the diagram back to step 1 and says, "*First*." And the procedure repeats itself with 6 and 12 as the two numbers.

All of this activity can be summarized in the following table:

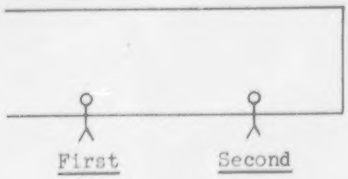
Control calls:	*First* writes or calls:	*Second* writes or calls:	Step:
	12	18	Begin
"*First*"	"No"		1
"*First* and *Second*"	18	12	2
"*First*"	6		3
"*First*"	"No"		4
"*First*"			1

Can you now begin to fill in the remaining responses in the table? What does First do at the point we stopped in the activity back at step 1? Does he call, "Yes" or "No"? ________. He must call, "No," since his number 6 is smaller than Second's number 12. Try completing the rest of the steps and compare with the completed table below.

Control calls:	First writes or calls:	Second writes or calls:	Step:
	12	18	Begin
"First"	"No"		1
"First and Second"	18	12	2
"First"	6		3
"First"	"No"		4
"First"	"No"		1
"First and Second"	12	6	2
"First"	6		3
"First"	"No"		4
* "First"	"Yes"		1
"First"	0		3
**"First"	"Yes"		4
			End

The last number that Second has written down is the greatest common divisor. As we already knew from our earlier examination, 6 is the greatest common divisor of 12 and 18.

What would First and Second have written on the chalkboard in front of them in finding the greatest common divisor of 12 and 18? See if you can fill in the space on the blackboard below:

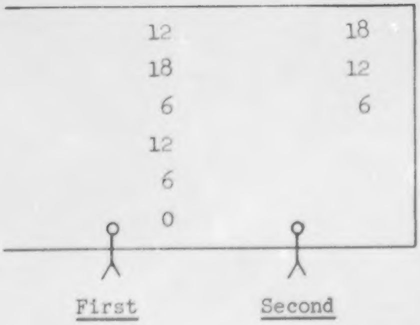


---

*At this point in the activity First says, "Yes," since 6 is not smaller than 6. Then the activity proceeds to step 3 since following the arrow skips step 2.

**At this point in the activity First says "Yes" since the difference is 0. This leads us to the end of the activity.

It should look like this:



Now try one on your own. Write out the table of responses for finding the greatest common divisor of 24 and 36.

Control calls:	First writes or calls:	Second writes or calls:	Step:
	24	36	Begin

You are correct if you find that the greatest common divisor of 24 and 36 is 12.

You can now set up your class to act as a computer in determining the greatest common divisor. This activity of the class illustrates several of the aspects of computers. In the first place, a relatively complex procedure is expressed by the repeated use of very simple procedures. In addition, good team work is necessary. Of course, the diagram is an illustration of simple computer programming.

CLASSROOM IDEAS

There are a number of ways in which this activity can be introduced in the classroom. If students have already learned how to find the greatest common divisor of two numbers, reference can be made to this. If students have not done this before, the activity might be introduced as the building of a computer to find the number needed to simplify a fraction such as $\frac{18}{24}$. The greatest common divisor of 18 and 24 is 6. Hence, the fraction can be rewritten as $\frac{3 \cdot 6}{4 \cdot 6} = \frac{3}{4}$. Regardless of what motivation is provided for the introduction of greatest common divisor, it might be appropriate to also introduce the computer by asking such questions as: "Have you ever seen a computer?", "Do computers have many parts or just a few parts?", "Can computers do complicated problems?", "How do computers do complicated problems?"

Obviously, Control will not be able to immediately take over his entire function. Control might be used to call out the students first while the teacher points to the appropriate step in the diagram and reads the steps. Control can be slowly worked into his whole job.

The diagram and the steps need to be on the chalkboard in advance. If an overhead projector is available, a transparency might be made of page 71. Control could then operate from the overhead projector rather than at the chalkboard.

Throughout the activity the class should be used to keep the computer operating smoothly. You must specifically plan to use the class or your attention may become directed solely to getting the computer to operate smoothly. The class should be particularly used to check the subtraction computation. Students may either handle the computation mentally or on the board or paper. Note that this activity is a good chance for subtraction drill.

A new group of students can be chosen to participate as computer parts. You might shift each time you present a new pair of numbers for determining the greatest common divisor. It might be appropriate to split the class into teams of three and let each group operate as a computer at the board or their seats. It is important to select initially a First who has a reasonable grasp of subtraction.

Naturally, the choice of pairs of numbers for which you find the greatest common divisor will lead to a variety of sequences of steps. Pairs such as 12, 18 and 24, 36 are problems which involve three subtractions. The pair 840, 378 involves eight subtractions, and the pair 35, 6 involves eleven subtractions. The pair 35, 6 has 1 as its greatest common divisor while the pair 30, 15 has one of the numbers of the pair as its greatest common divisor. Incidentally, the pair 30, 15 involves only two subtractions. They don't come much easier than that. The examples you choose will depend upon what you wish to emphasize with your class.

OUTLINE

Below is an outline which you can use to demonstrate in your classroom the computer activity for finding the greatest common divisor of two numbers:

1. Display the diagram and steps.
2. Introduce greatest common divisor.
3. Introduce computers.
4. Select students for computer parts.
5. Select pair of numbers for finding greatest common divisor.
6. Instruct First and Second to write their numbers on board.
7. Point to first step in diagram.
8. Instruct Control to call student.
9. Read the instructions for the student.
10. Continue through steps of activity letting Control take over your job and using the class as a check on students.

OBJECTIVES

As minimum objectives for the students at the completion of the classroom activity with the computer, you might expect:

1. The student will use repeated subtraction to find the greatest common divisor of any two numbers.

2. The student will demonstrate the operation of the computer by playing the role of Control in the computer activity.

Naturally, one of the longer range goals would be the expectation that students will play the role of Control in another computer program activity. Ultimately, the student will write his own diagrams and steps for procedures in mathematics.

APPRAISAL

You can determine if the students have acquired the immediate behaviors for the activity described in the objectives by asking the following questions. Questions one and two can be asked of all students as an exercise to be written; question three will require a demonstration with students.

(Obj. 1) 1. "Using the method of the computer, can you find the greatest common divisor of 8 and 6?"

(Obj. 1) 2. "Using the method of the computer, can you find the greatest common divisor of 42 and 70?"

(Obj. 2) 3. Select a First and Second. Then choose the student to be evaluated and ask, "Can you demonstrate how you would be Control for the computer in finding the greatest common divisor of 21 and 14?"

LINEAR FUNCTION GAMES

(Prepared by W. G. Lister and J. Latino)

GENERAL DESCRIPTION OF VARIOUS LINEAR FUNCTION GAMES

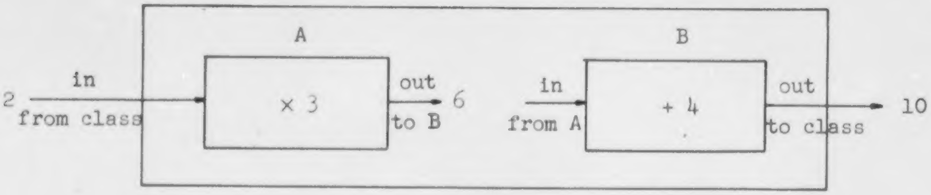
A variety of classroom activities can be devised in which all pupils can participate either as simple arithmetic "machines" or as analysts of the operation of "assemblies" of such machines. The cluster of games described below uses the basic arithmetic transformations like "multiply by 3" or "add 4" as simple machines.

The fundamental elements of these games are the following:

1. A pupil, the machine, selects, or is assigned by the teacher, one of an agreed upon set of rules for transferring or converting a number into another number. The rule might be, for example, "multiply by 3."
2. A member of the class selects a number, the input, which is communicated to the machine.
3. The machine announces the number, the output, into which his rule converts the input. Thus, in the example cited, if 5 is the input he announced "15" as the output.
4. The "analysts," i.e., the class, now try to identify the operation of the machine. In the example cited above this is very simple and if the machine is known to be a multiplier, one input will always suffice. In slightly more complex arrangements the situation is more interesting, however.

Suppose that, in place of the simple machine described in 1. above a "hook-up" of two simple machines is constructed. One pupil A is an unknown multiplier while a second B is an unknown adder, i.e., he adds a fixed number to each input to obtain the associated output. The input number is fed to A who communicates, privately, his output to B. The latter takes this as input and announces the final output to the analysts, the class.

For example:



Thus in this case the compound hook-up converts a 2 input to a 10 output. Similarly, if 1 is fed into the machine 7 comes out and if 5 is fed into the machine 19 comes out.

The problem in 4. for the analysts is to identify what each component, A and B, is doing. A number of third grade students are capable of discovering quickly that the hook-up can be identified by testing it with any 2 inputs. Some will see almost immediately that a zero input gives away the adder, B.

SUGGESTED GAMES

SINGLE COMPONENT GAMES - Grades 1 through 6

Function Game 1

In introducing the game the teacher can play the role of the machine. Game 1 is a simple "adder" game. The teacher announces that she is a machine which adds the same amount to each number that is given to her. The children are asked to suggest inputs. The teacher then silently operates on the numbers and reveals the results in the form of an output. It would be advisable to list the set of inputs and outputs on the board as follows:

Inputs		Outputs
1	→	4
2	→	5
3	→	6

From the record of inputs and outputs it is clear that the teacher is adding 3 to each input. Except for grade one most children need only one input to decide what a single component machine is doing. After the teacher has played the game several times, a student would be asked to play the role of the machine. Have the student sit in a desk situated in the front of the room and facing the class. Select another student who will record the inputs and outputs on the board. After the class has participated in concert, pair off members of the class. One person in each pair now serves as a machine. The other person feeds inputs into the machine and records them on paper. When he has determined the number the machine is adding, he reverses roles and plays the part of the machine. In the upper grades single component machines that are "adders" will be of only momentary interest.

Function Game 2

This single component game is identical to game 1 except that the component is a multiplier. Thus, the game could be played in grades two through six with the same class organization as suggested in game 1. Again, the students will quickly see that one input will determine what the machine is doing. These games will be short but they are necessary to develop knowledge of the behavior of machines necessary for the more complicated games.

Function Game 3.

Once the children have had experience with machines that add a number and machines that multiply by a given number, the teacher should play the following game:

Announce that the machine may be an "adder" or a "multiplier." The teacher acts as the machine and asks a student to act as recorder of inputs and outputs on the board. Each student is also asked to keep a record of inputs and outputs on paper. As before, the class feeds inputs into the machine and seeks to determine what the machine is doing. This game would be suitable for grades three through six.

Suppose the children feed 2 into the machine.

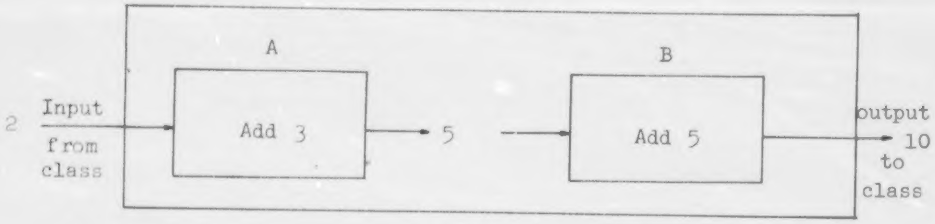
input 2 → output 4

The children should soon see that one input is not sufficient for determining what the machine is doing. It could be adding 2 or multiplying by 2. A second input, say 3, yields output 6. Now it is clear the machine is multiplying by 2 and not just adding 2. Once the children have practiced the game pair off as in game 1.

GAMES INVOLVING TWO OR MORE COMPONENTS - Grades 1 through 6

Function Game 4

With this game we start "hooking up" two or more components. This particular game is restricted to components of the same type. Suppose two children are privately assigned instructions as follows: "A" adds 3 and "B" adds 5. Hooked up in a series they form a two-component machine. Have both students sit in desks facing the class. The desks must be close enough for student A to whisper his output to student B. The rest of the class keeps a record of inputs and outputs on paper at their desks. It is advisable for the teacher to keep a record at her desk.

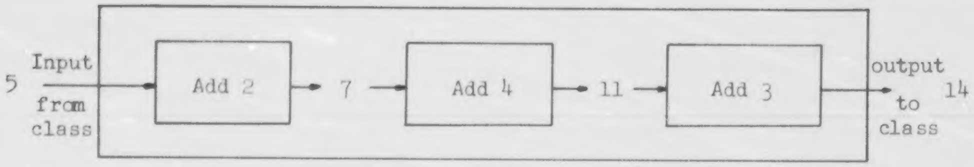


The class now feeds inputs into the machine. Since A secretely feeds his output to B, the class again knows only the final output from B for each input from A. Here is a possible table of inputs and outputs:

Inputs	Outputs
1	9
2	10
3	11

The children will discover that the two-component machine can be replaced with a one-component machine that "adds 8." This is as interesting a way to note that an addition of 3 and an addition of 5 is equivalent to an addition of 8 as usual column addition.

Continue the game with three adders. Allow each child to choose an adder that he keeps secret. The class then chooses three students and decides on their order. The three students now are sitting in front of the class forming a three-component machine. Proceed as before with individual records kept at desks. For example:

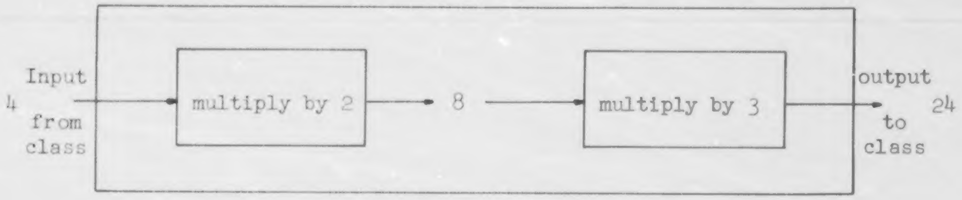

5
Input
from
class
Add 2
7
Add 4
11
Add 3
output
to
class
14

The equivalent component would be a part that adds 9.

String as many adders as you find feasible together. The children enjoy this type of participation and soon conclude that any sequence of "adders" is replaceable by a single "adder."

Function Game 5

This game is identical to function game 4 except that the components are all "multipliers." Since the children should be accustomed to operating the machines, break the class into groups of about six each and instruct each group to play the game alternating their roles as analysts or machine. Generally the children should be keeping their own records of inputs and outputs. As the game proceeds there may arise situations which would interest the entire class. In that event have the class all work on one machine and act as recorder and stimulator. For example:


Input
4
from
class
multiply by 2
8
multiply by 3
output
to
class
24

Inputs	Outputs
0	0
1	6
2	12
3	18

As with the "adder" only machine, the "multiplier" only machine is replaceable with a single component machine, in this case by a component which multiplies by 6.

If we "hook-up" three factors (multipliers) in a row, the children again note that a single component will replace the three factors. For example, the factors 3, 2, 5 may be replaced with the factor 30.

Experience has shown that it is more exciting and more meaningful for children to discover the properties of addition and multiplication in this manner than by learning phrases such as "commutative" and "associative." Here they have an opportunity literally to move the factors (components) around and find out if their order affects the output. Thus in games 4 and 5 the teacher should encourage the children to experiment with the multi-component machines by moving the individual components around and note what effect if any the new positions have on the output. This activity is best directed by the teacher. Have the class all operate on one machine in front of the class.

While game 4, which concerns only addition, can be started as early as grade 1, clearly game 5 involving multiplication cannot be started before grade 3 with most groups.

The multi-component factor machines afford many interesting situations. The multiplier "1" is soon noted to leave the input unaffected. Select three children who are assigned the factors 2, 3, and 5. They sit facing the class. The teacher directs the questioning. If we use only prime multipliers such as 2, 3, 5 then a single multiplier of 30 will replace the three components. But we can ask the class what the three components must be if none of the components is "multiply by 1." The class must respond with 2, 3, 5. But suppose 4 and 6 are multipliers in a two-component machine. Again, assume that "1" is ruled out as a component. Since the children cannot know the nature of each component, they see at first that a single component "multiply by 24" can replace the two components. Ensuing discussion about the nature of the two components should reveal that several pairs are possible, namely, 12 and 2, 8 and 3 beside 4 and 6. Thus they cannot uniquely determine the two components unless the single multiplier is the product of two prime factors. This can lead to interesting speculation about the classification of the integers into composites and primes and the special nature of "1."

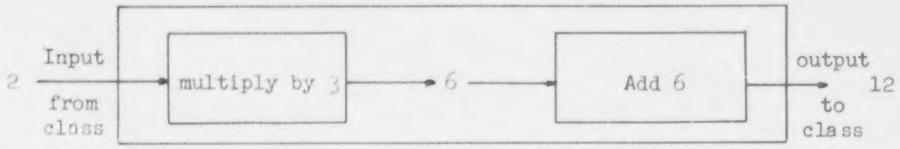
Function Game 6

This is a further development of multi-component games that have only one operation. In this case do not indicate to the class whether the components are "adders" or "multipliers." Now more than one input is needed to determine whether the machine is adding or multiplying. The rest of the game proceeds as in games 4 and 5.

GAMES INVOLVING TWO COMPONENTS AND TWO OPERATIONS - Grades 3 through 6

Function Game 7

In this game the children are told that the first component is a multiplier and the second component is an adder. For example, multiply by 3 and then add 6. Two students sit facing the class and act as the machine. The teacher keeps the record on the board.



Inputs	Outputs
0	6
1	9
2	12
3	15

The children have to find out what multiplier is being used and what adder is being used. Third grade children have shown a quick realization that the input zero reveals the adder. They also seem to sense that when inputs "one apart" are fed into the machine, the difference of the outputs is the multiplier. This game should be played until the children are ready for complications. Third, fourth, fifth and sixth grade children find the game interesting and informative. Allow each child a chance to think of a two component machine that multiplies and then adds. Then select a student and have him sit in front of the class acting as a two component machine. The children keep records at their desks. Continue to select different students once the make-up of a machine is correctly discerned. Now assign one part of the class to think of a "multiplier" and the other part of the class to think of an "adder." Have the children select one of the multipliers and one of the adders who advance to the front of the room where the game continues as before.

A great many variations on this fundamental theme can be devised to suit various experience and maturity levels.

Function Game 8

In this game we permit a "hook-up" in which it is known that one component is an adder and the other a multiplier, but it is not known which component operates first. In this case it is generally not possible to identify which component is the adder and which is the multiplier for there are two "hook-ups" with the same final effect. Assign in private an adder and a multiplier to two children seated in the front of the room. Records are kept at the desks. For example:

Suppose the machine containing an adder and a multiplier responds as follows:

Input	Outputs
0	6
1	8
2	10
3	12

The machine could be multiplying by 2 and then adding 6 (the more likely choice by the class). The machine could be adding 3 and then multiplying by 2.

It is good technique to indicate that the machine is doing the second when the class chooses the first. Both are obviously equivalent. This is an excellent way to get at "distributivity." Children are intrigued to find out when and why the adding-multiplying machines have two descriptions. This is also a natural way to introduce rational numbers as adders or multipliers. (see Game 10.)

Function Game 9

Encourage the children to reverse the order of the adder and multiplier. What effect does this have on the outputs, if any?

For example:

in → $\boxed{\text{multiply by 2}}$ → $\boxed{\text{add 3}}$ → out

Inputs	Outputs
0	3
1	5
2	7
3	9

in → $\boxed{\text{add 3}}$ → $\boxed{\text{multiply by 2}}$ → out

Inputs	Outputs
0	6
1	8
2	10
3	12

Clearly the machine behaves differently.

Have the class consider the questions: What multipliers and what adders are reversible so that the machine yields the same outputs? For example:

multiply by 1 add 7

Inputs	Outputs
0	7
1	8
2	9
3	10

add 7 multiply by 1

Inputs	Outputs
0	7
1	8
2	9
3	10

Now the reversed machine behaves in the same manner as the first machine. Of course, the multiplier "1" is the key. The adder is immaterial.

Function Game 10

For classes that have considered subtraction and division game 10 is revealing. Game 10 has also been used to introduce the necessity for inventing a division operation. In any case, children find this variation of the two operation, two-component machine very instructive.

Present an adder-multiplier machine and have the children try to discover what the multiplier is and what the adder is. In this game they know the multiplier is first and the adder is second. Proceed as in game 7.

For example:

in → multiplier → adder → out

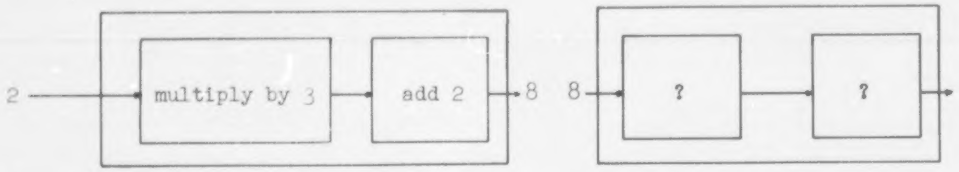
Inputs	Outputs
0	2
1	5
2	8
3	11

The children should guess that the machine is multiplying by 3 and then adding 2.

The point to game 10 is to find a machine (two components) which will take the set of outputs from the first machine and convert them back into the original inputs. To facilitate understanding of what we are after place two more students next to the two students who are acting as the original multiplier-adder machine. Also place the diagram indicated below on the board.

For example: input 2 into the machine above yielded output 8.

We want a new machine such that when 8 is put into the machine, 2 comes out. In essence, we want an "inverse" machine. In diagram form:



Draw these on the board with explanation.

Inputs	Outputs	Inputs	Outputs
0	2	2	0
1	5	5	1
2	8	8	2
3	11	11	3

Children soon sense that some type of "opposite" operation is necessary. In the example above, subtract 2 and divide by 3 would do the job. This is a fine way to deal with inverse operations. The children find it more dynamic and meaningful.

In the fifth and sixth grades multiplication of rational numbers will make possible multiplication by $\frac{1}{3}$ in lieu of division by 3. Thus, the inverse machine for the machine "multiply by 3 then add 2" would be "subtract 2 then multiply by $\frac{1}{3}$."

Game 10 opens up a whole range of experiences with operations and rational numbers. For grades 4 through 6 the following are suggestive of this range.

Example 1.

Choose two children and indicate that the first is a multiplier and the second is an adder. The children are to find what each component is doing and a reverse machine.

Inputs	Outputs
0	3
2	4
4	5
6	6

The children should indicate that the first child is multiplying by $\frac{1}{2}$ and the second child is adding 3. The reverse machine would be "subtract 3 and multiply by 2." This raises the question of reciprocals in a nice manner.

Example 2.

Choose two children and indicate that the first is a multiplier and the second is a subtractor. (This can lead to negative integers or they can be avoided through judicious choice of inputs. Experience has shown that children can handle the negative integers quite easily).

Inputs	Outputs
0	-1
2	2
4	5
6	8

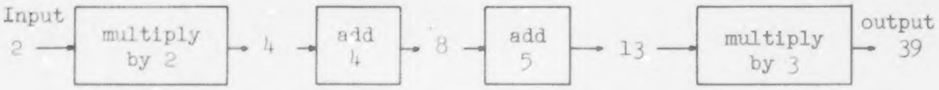
The children should indicate that the first child is multiplying by $\frac{3}{2}$ or $1\frac{1}{2}$ and the second child is subtracting 1. The reverse machine would be "add 1 and multiply by $\frac{2}{3}$."

GAMES INVOLVING MORE THAN TWO COMPONENTS AND TWO OPERATIONS - Grades 3 through 6

A good practice in all the above games and especially in the ensuing game is to designate in secret four or five children as multipliers and four or five children as adders. Then the rest of the class may choose a component from the multiplier group and a component from the adder group. Game 11 would have them select more than one from each group.

Function Game 11

Assume the class has selected two multipliers and two adders. Have the selected students sitting in front of the class and facing the class. All other students keep records. Although the factors and addends would be unknown to the class, let us assume they are 2 and 3 from the multipliers and 5 and 4 from the adders. Allow the class to assemble those four components in any order. Below we have assumed only one of several orders.



Inputs	Outputs
0	27
1	33
2	39
3	45

The children soon find out that a two-component machine made up of a multiplier followed by an adder can replace the four components. In this instance "multiply by 6 and add 27" would suffice.

The question game 11 raises occasions much lively debate among intermediate grade children. Will a six-component machine containing multipliers and adders be replaceable with a two-component machine? Will a five- or seven-component machine be replaceable with two components? Of course, it turns out that any combination of adders and multipliers can be replaced with one adder and one multiplier.

Give the children free rein in hooking up as many components as they desire. It is a nice challenge to their arithmetic prowess to find the two-component machine that will serve as a replacement.

THE MATHEMATICS OF LINEAR OPERATORS

In the activities described above each pupil personifies a simple correspondence scheme or function. The output is the number which in the given scheme corresponds to the input. Thus "multiply by 3" can be identified with the function described by $n \rightarrow 3n$, a convenient code for recording the general relation of output $(3n)$ to input (n). Similarly,

$$m \rightarrow m + 4$$

denotes the "add 4" function.

Such functions can be joined in series to form a new function whose correspondence scheme consists of the first function in the series followed by the second. Thus,

$$n \rightarrow 3n \rightarrow 3n + 4$$

reveals that the function $n \rightarrow 3n + 4$ can be analyzed as $n \rightarrow 3n$ followed by $m \rightarrow m + 4$. Also, $n \rightarrow 2n \rightarrow 3(2n)$ shows that $n \rightarrow 2n$ followed by $m \rightarrow 3m$ yields $n \rightarrow 6n$.

Functions which can be described as a multiplier followed by an adder are called linear functions. These include multipliers (make the adder 0) and adders (make the multiplier 1).

Reversing the component parts of a linear function generally gives a new function. Thus

$$n \rightarrow 3n \rightarrow 3n + 6$$

reverses to $n \rightarrow n + 6 \rightarrow 3(n + 6) = 3n + 18.$

Notice, however, that the distributive property assures that an adder followed by a multiplier is again a linear function. Thus if the order of adder and multiplier is unknown the components cannot be determined from the final input-output correspondence. For example:

$$n \rightarrow 3n \rightarrow 3n + 18$$

and $n \rightarrow n + 6 \rightarrow 3(n + 6)$

represent two different hook-ups which do the same job.

The basic game depends heavily on the fact that any linear function is completely determined by two input-output pairs. For instance, if we know that

$$2 \rightarrow 12 \quad \text{and} \quad 4 \rightarrow 18$$

and if we observe that an adder makes no contribution to the difference in outputs (12 to 18) then we conclude that the multiplier is 3 and hence, the adder is 6. The argument is evident if we write

$$(3m + 6) - (3n + 6) = 3(m - n)$$

Output No. 1 - Output No. 2 = (multiplier)(difference in inputs).

This argument also shows that if two input-output pairs are specified in advance, a linear function which effects them can be found provided rational numbers are allowed and subtraction or use of negative adders is permitted. For example, to achieve

$$6 \to 8 \quad \text{and} \quad 4 \to 5$$

using a multiplier followed by an adder, the multiplier must be $(8 - 5) \times (6 - 4) = \frac{3}{2}$ and the adder $8 - (\frac{3}{2})6 = -1$ giving the function $n \to (\frac{3}{2})n - 1$.

Machines made of two or more component adders of multipliers reflect the properties of these operations. The fact that reversing the order of two adders does not alter the final result is a consequence of the independence of order for summands:

$$\left\{\begin{matrix} n \to n + 3 \to (n + 3) + 4 \\ n \to n + 4 \to (n + 4) + 3 \end{matrix}\right\}$$

The fact that any chain comprised of a mixture of adders and multipliers can be replaced as far as final effect is concerned by a single multiplier followed by a single adder is a consequence of the fact that any part consisting of adder followed by multiplier can be replaced by a multiplier followed by an adder.

Thus,

$$n \to 3n \to 3n + 5 \to 4(3n + 5) \to 4(3n + 5) - 6$$

can be rebuilt as

$$n \to 3n \to 12n \to 12n + 20 \to (12n + 20) - 6$$

and this in turn can be reduced to

$$n \to 12n \to 12n + 14.$$

In this case we rearranged

$$\overset{3}{\text{multiplier}} \to \overset{5}{\text{adder}} \to \overset{4}{\text{multiplier}} \to \overset{-6}{\text{adder}}$$

first to

$$\overset{3}{\text{multiplier}} \to \overset{4}{\text{multiplier}} \to \overset{20}{\text{adder}} \to \overset{-6}{\text{adder}}$$

then to

$$\overset{12}{\text{multiplier}} \to \overset{14}{\text{adder}}.$$

This means an analysis also shows why and how it is possible to find an inverse for a linear function, that is a function which has inputs and outputs reversed.

The inverse of $n \to 3n + 5$ must have

$$8 \to 1 \quad 14 \to 3 \quad 23 \to 6$$

and in general must convert $3n + 5$ back to n. This can be achieved in steps, as follows:

$$3n + 5 \to (3n + 5) - 5 \to \frac{1}{3}(3n) = n.$$

We simply undo each component from back to front.

$$\begin{tikzcd}[column sep=huge] n \arrow[r, shift left, "\text{3 multiplier}"] & 3n \arrow[l, shift left, "\text{multiply } \frac{1}{3}"] \arrow[r, shift left, "\text{5 adder}"] & 3n + 5 \arrow[l, shift left, "\text{subtract 5}"] \end{tikzcd}$$

Thus, $m \to \frac{1}{3}(m - 5) = \frac{1}{3}m - \frac{5}{3}$ is the inverse of $n \to 3n + 5$ as can be checked again by following either with the other.

$$m \to \frac{1}{3}m - \frac{5}{3} \to 3(\frac{1}{3}m - \frac{5}{3}) + 5 = m$$

The point here is that a pair of functions have inputs and outputs reversed provided that when either is followed by the other the final output always coincides with the input, i.e., $m \to m$.

OBJECTIVES, USES AND EXPECTED VALUES OF LINEAR FUNCTION GAMES

The present elementary mathematics curriculum tends toward formalism in the manner in which children are introduced to the properties of operations. We even have some classrooms in which children are learning to spell "commutative" in second grade. Linear function games are an excellent vehicle for allowing the child to experience the properties of the fundamental operations in a more exciting and instructive setting. The use of children as components makes the operations come "alive." Children sense from the use of machines that:

1. addition is commutative and associative
2. multiplication is commutative and associative
3. multiplication is distributive over addition
4. addition is not distributive over multiplication
5. zero and one have special properties.

All this without the questionable verbage that has become associated with reform in the elementary mathematics curriculum.

The notion of equivalent statements comes through very clearly. Children argue and experiment and seem to feel equivalence much better through the use of a device that is physical and as dynamic as a machine, albeit a human one.

Aside from the very valid mathematical insights such machines afford, (the notion of a function, inverse function, inverse operations, etc.) the linear function machines serve the real need of loosening up the child's approach to the operations.

There are some very practical ends served, namely:

1. Drill in arithmetic occurs naturally and intensively.
2. Children must master simple arithmetic to operate the machines effectively. The game motivation seems to work better than a mark motivation.
3. Children who are not successful in paper and pencil arithmetic have a better chance with the machines either as components or as analysts.

In terms of philosophy of instruction the use of games and functions machines reflect the need to allow the child to guess, think and then induce some sort of hypothesis and then conclusion. The amount of "if-then" thinking that transpires during the playing of linear function games is impressive.

The mathematical experience for children is certainly different than the feedback variety of early arithmetic instruction. For this reason alone linear function games deserve a place in the grade school.

Linear function games have:

1. Led a third grade class to invent division (in search of inverse machines).
2. Led a fourth grade class to invent negative integers.
3. Led to a discussion of reciprocals in a fourth grade class.
4. Led to a discussion of operations on rationals in general in a fifth grade class.
5. Led a second grade class to consider negative integers.

Finally, linear function games have infused into many classrooms some life into a subject which was becoming performance on a piece of paper.

GAMES WITH ADDITION TABLES

(Prepared by W. G. Lister and J. Latino)

These games depend on a child's ability to deal with the addition table. The important aims of the games are:

1. Promote familiarity with the properties of the operation.
2. Note specific relationships.
3. Practice in number facts.
4. Allow the student to make up some puzzles of his own.
5. Allow some opportunity to make inferences from experimental data.

Addition Game 1

Classroom experience indicates that it is best to motivate the general topic of addition table games with the ensuing game.

This might be called number "bingo." The teacher prepares a 6 x 6 table as follows:

⑦	15	13	10	8	14	13
②	10	8	5	3	9	8
⑤	13	11	8	6	12	11
④	12	10	7	5	11	10
⑨	17	15	12	10	16	15
①	9	7	4	2	8	7
+	⑧	⑥	③	①	⑦	⑥

Then remove the marginal entries and make the following on a ditto:

15	13	10	8	14	13
10	8	5	3	9	8
13	11	8	6	12	11
12	10	7	5	11	10
17	15	12	10	16	15
9	7	4	2	8	7

Each child now has a table. The teacher directs a playing of the game.

Rules: The children can select a number from a given column and a given row only once. Thus, if a number is selected from row 1, no other number from row 1 may be selected. Similarly, if a number is selected from column 3 no other number may be selected from column 3. In a 4 by 4 table this would limit the choice to 4 numbers. In a 5 by 5 table the limit would be five numbers, etc.

COLUMNS						Rows
1	2	3	4	5	6	
				14		1
		5				2
13						3
					10	4
	15					5
			2			6

The teacher writes a number on a piece of paper before the choices are made. Of course, the teacher chooses as the number the sum of all the entries in the margin which now do not appear. It should be clear that the rules of the game force the children to select numbers whose total equals the total of the marginal entries. After the children have made their choices, the teacher asks the child to whom she has given the paper bearing the number to read the number. The children are now quite impressed. For the table shown to the left, the choices would always total 59.

Let us presume that the children choose from the above table as follows:

1. From column 1, number 13. (Columns are numbered from left to right). They cannot choose anymore from that column or the row in which 13 resides.
2. From column 2, number 15. The children cannot choose anymore from column 2 or row 5 where 15 resides (Rows are numbered from the top, down.)
3. In a similar manner 5 is chosen from column 3; 2 from column 4; 14 from column 5; and 10 from column 6 (the only choice left).

Note that the sum of the choices is 59. Of course, the choices do not have to be made in any order. A selection from column 4 or row 3 could be first, for example. As long as the rules are followed the total of the choices will always be 59 for this particular table.

The random appearance of the numbers tends to make the game very interesting. The teacher should make several "bingo" dittoes with various addition tables in which the marginal entries have been removed. Various plays of this game will motivate an investigation of addition tables in general. The teacher should not reveal that the table is an addition table.

Addition Game 2 -- Grade level -(2-6)

In order to play this game the children must be acquainted with the format for addition tables. A week or so prior to the playing of the game, the children should do some 2 × 2 or 3 × 3 addition tables.

Examples:

④	5	9
③	4	8
+	①	⑤

or

③	5	9
②	4	8
+	②	⑥

or

⑦	12	9
④	9	6
+	⑤	②

After a suitable lapse of time (usually about a week) introduce the following game.

Draw a blank 2 × 2 table on the board with no provision for marginal entries.

Thus:

Now ask the class to choose a number and a box in which the number is to be written.

Example: Children first choose 4.

	4

Now ask the class to choose a second number and a box for its location. Assume these choose 7.

	4
	7

Now ask the class to choose a third number and box. Assume they choose 6.

6	4
	7

The teacher chooses the final number and places it in the remaining box. Her choice is made so that the diagonals have equal totals, that is $4 + 9 = 7 + 6$.

6	4
9	7

9 is chosen by the teacher to make the diagonals equal.

Play this game at least four or five times leaving the various tables made on the board. Hand out a ditto to the students which contains six blank 2 × 2 table forms. Have them record the results.

Ask the class if they can guess how the final choice is being made. Generally, they respond in terms of constant differences in the columns or rows. Thus in the example above the student often claims that the 9 was chosen in order that it be 2 more than 7 since 6 was two more than 4. Alternatively, the 9 was chosen in order that it be 3 more than 6 because 7 was 3 more than 4. They do not realize at this time that those conditions are equivalent to making the diagonals equal.

Now the teacher makes circles in the margins of the existing tables and asks the class if they can fill in the marginal entries to fit the number in the body of the table.

Example:

②	6	4
○	9	7
+	○	○

One possible set of marginal entries would be:

②	6	4
⑤	9	7
+	④	②

Another set of entries would be:

①	6	4
④	9	7
+	⑤	③

Another set would be:

③	6	4
⑥	9	7
+	③	①

The children are quite interested in the fact that there are several possibilities for the table. (In fact, with the negative integers, there are infinitely many.)

The teacher should try to evoke from the class some of the following:

1. A 2 × 2 table with equal diagonal sums will always make an addition table.
2. A table with equal diagonal sums has a constant row difference and a constant column difference. (2 × 2 table)
3. The sum of the four entries in a 2 × 2 table with equal diagonal sums is always an even number.

At this juncture it is time to give the children a counter-example. Place on the board a 2 × 2 table whose entries are such that the diagonals have different sums.

For example:

○	7	6
○	4	5
+	○	○

Now ask the children to make entries in the margin for an addition table.

(Cannot be done.)

There are some interesting comments that usually ensue. The conclusion usually is that a necessary and sufficient condition that a 2 × 2 table be an addition table is that the column or row differences be constant or that the diagonal sums be equal.

Variations on Addition Game 2

Once the children have become accustomed to game 2 start to explore different schemes for placing entries in the bodies of the table.

Have the children choose 4 consecutive integers as entries. Thus,

○	3	4
○	5	6
+	○	○

Is there a set of numbers for the margin which will make it an addition table?

Have them try consecutive even integers.

4	6
8	10

Or consecutive odd integers.

9	11
13	15

Have the children try multiples of a given integer.

For example: Multiples of 3

6	9
12	15

Multiples of 7

7	14
21	28

Have the children try numbers with a constant difference.

For example: 9, 13, 17, 21

9	13
17	21

Or, 24, 34, 44, 54

24	34
44	54

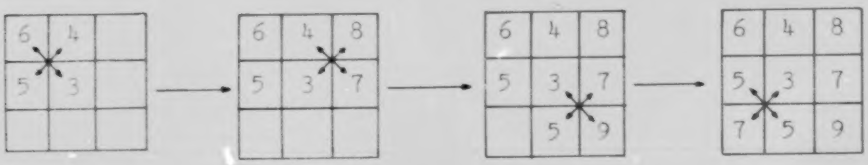
Of course, in all of the above cases the numbers are entered in order from left to right as indicated.

See if the children can discover the ONCE A NUMERAL IS CHOSEN FOR THE MARGIN, ALL THE OTHER MARGINAL ENTRIES ARE ESTABLISHED. There is a nice bit of "if-then" thinking involved here.

Addition Game 3

We are now prepared to extend our tabular work to 3 × 3, 4 × 4, etc., tables.

We can regard any table of higher order than 2 × 2 as an extension of a 2 × 2 table in the following way: Fill one 2 × 2 corner of a 3 × 3 so that the diagonals are equal. Then fill the adjacent corner in the same way except that two entries are already determined. Continue for the rest of the table.



Once the 3 × 3 table is filled have the children try to find marginal entries which will make an addition table.

Speculation should now center about the following questions:

1. Will there always be an addition table when the body of the table is filled in this way?
2. What is the sum of the nine numbers in the table? What is the sum of the marginal entries?
3. Are there 3 × 3 tables that have one and only one way to fill in the margins?
4. Are the diagonals always equal in 3 × 3 addition tables?
5. Can a 3 × 3 table be made with equal diagonal sums which is not an incomplete addition table?

The teacher should now have on hand ditto forms containing 4 × 4, 5 × 5 6 × 6, 7 × 7, 8 × 8 and 9 × 9 matrices. Encourage the children to experiment with entries in the body of the table that will result in an addition table.

Finally, see if the children can discover the trick to the original game which the teacher played. How did the teacher know what the sum of the choices would be?

It is clear that the above games provide for much drill in addition to an interesting setting. The games have been played with children in grades two through eight. Even some high school classes have found the games fascinating. It is impossible to list all the types of questions that arise during plays of the games. One thing is evident, however. This is one of the best vehicles for speculation about the nature of addition and the properties of number that the writers have employed.

Parenthetically, rational numbers may be used in all the above games.

GAMES WITH MULTIPLICATION TABLES

In pages 97 through 103 we have indicated the possible uses to which the addition table may be put. There are obvious extensions of this same scheme to multiplication tables. But multiplication tables involve greater difficulty if rational numbers are to be avoided. That is, generally we think of a multiplication table as having integral marginal entries. But the difficulties also make for some challenging problems for the children.

Multiplication Table Game 1

We start with a 2 × 2 multiplication situation. Since the children should be fully accustomed to tabular games, the introduction can be brief.

The teacher places a 2 × 2 matrix on the board and makes entries so that the products of the diagonals are equal.

For example:

8	6
20	15

Now the class is asked if there is a multiplication table that fits the given table. That is, are there INTEGRAL ENTRIES for the margin?

It will be immediately clear that the children do not have the same freedom in filling in the marginal entries that they had in the addition table. The stipulation that we have integers in the margin limits the marginal possibilities.

A possible margin for the above table would be:

2	8	6
5	20	15
×	4	3

Is there another possibility?

1	8	6
	20	15
×	8	

8 is not a factor of 20.

In attempting to find another set of marginal entries for the table at the left, the child must consider such statements as: For the row containing 8 and 6 as products, there are only two possible entries for the margin, namely, 1 and 2. "One" cannot work since that would require 8 to be the entry for the first column and 8 is not a factor of 20. Thus there is only one set of marginal entries for this table. (We remind you that the entries are to be positive integers.)

Have the children consider tables in which there are more than one set of possible marginal entries.

For example:

4	32	40
6	48	60
×	8	10

or

2	32	40
3	48	60
×	16	20

or

8	32	40
12	48	60
×	4	5

The children should consider the following questions:

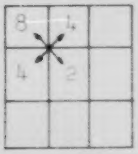
1. What is the product of each of the diagonals?
2. What is the product of the four marginal entries?
3. If we make the diagonal products equal, will there be at least one set of marginal entries that will make a multiplication table?
4. For those classes which are acquainted with primes, consider tables with only prime entries in the margin.
5. Is it possible to get a 2 × 2 multiplication table with only primes in the body of the table? (Note, "One" is not a prime.)
6. For a table which contains only primes in the body of the table, what marginal entry is always necessary?

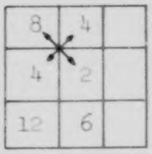
Multiplication Table Game 2

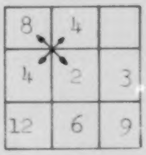
As with the addition tables we now encourage the children to extend their multiplication table work to 3 × 3 tables, 4 × 4 tables, etc.

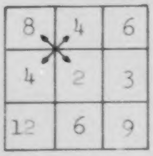
If we follow the pattern noticed in the addition tables, that is, making adjacent diagonals have the same product, will it always be possible to find marginal entries for a multiplication table?

For example:





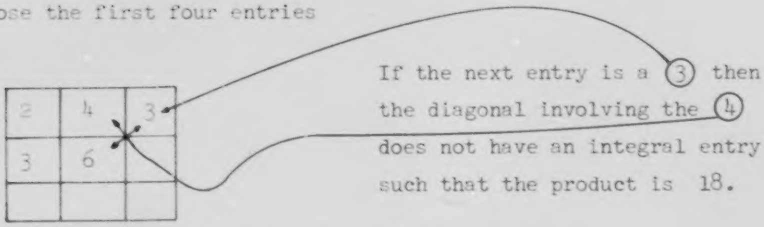




After exploring this procedure the teacher should provide dittoed matrices of 3 × 3 and 4 × 4 tables and allow the children to experiment by filling the body of the table and searching for marginal entries. There are pitfalls here that make for interesting speculation about the nature of factors.

For example:

Suppose the first four entries are:



If the next entry is a ③ then the diagonal involving the ④ does not have an integral entry such that the product is 18.

The class should consider the following questions:

1. What is the product of a diagonal of a 3 × 3 table? What is the product of the marginal entries?
2. Are there 3 × 3 tables for which there is only one set of marginal entries?
3. Are there 3 × 3 tables for which the entries in the body are all primes?

Multiplication Table Game 3

Recall that we introduced tabular games with our number "bingo" games. The rules for this game were: (1) From a given table numbers are to be selected. (2) Only one number may be selected from a given row and a given column. (3) The product of the numbers selected is written on a piece of paper BEFORE THE SELECTIONS. (4) The class is to ascertain how the number could be predicted ahead of time.

For example:

For the table

6	9	3	6
2	3	1	2
4	6	2	4
8	12	4	8

Assume the class selects the circled numbers.

6	9	③	6
②	3	1	2
4	6	2	④
8	⑫	4	8

The product of the selections is 288 every time.

This is clear when we make the table with its marginal entries. The product of the marginal entries is 288.

③	6	9	3	6
①	2	3	1	2
②	4	6	2	4
④	8	12	4	8
×	②	③	①	②

The teacher should make dittoes containing tables without marginal entries and play the game with the students. Once the students have learned the game, they should be encouraged to make "bingo" tables by themselves.

Multiplication Table Game 4

An interesting variation of the use of the multiplication table involves "incomplete" tables. The child is given a table with only certain entries in the body of the table. He is to complete the table. This can be easy or difficult depending on the information given. The teacher should construct many incomplete tables and present them to the class for completion at their leisure.

Examples:

	16
6	

	8	
2		
		6

2		
		3
		5

	6	
7		4
	9	

0	1	
		1
	0	

Start with 2 × 2 because they are quite easy. But even a 3 × 3 table can require much trial and error and thought about factors. Remind the students that only positive integers or zero are acceptable.

After the students have tried those incomplete tables made by the teacher, they should be encouraged to make some of their own. Students like to present incomplete tables to the class in an attempt to "stump" the rest of the class. Classroom experience has produced some interesting pupil-made incomplete tables.

Multiplication Table Game 5

Another variation involves table entries made by starting with any given number and using a constant factor to generate the entries which are recorded in sequence from left to right.

Example: 2, 6, 18, 54 and four entries for a 2 × 2 table. The "2" was selected at random. The 6, 18, and 54 were obtained by multiplying successively by 3, that is, 3 × 2 = 6, 3 × 6 = 18, 3 × 18 = 54.

Column 1	Column 2	Row
2	6	1
18	54	2

$$\frac{2}{18} = \frac{6}{54}$$

or

$$\frac{2}{6} = \frac{18}{54}$$

In their initial work with 2 × 2 multiplication tables, the children might not have noted that the ratio of the numbers in column 1 to the numbers in column 2 is constant. Also the ratio of the numbers in row 1 to the numbers in row 2 is constant. By the fifth or sixth grade the teacher can encourage looking for the ratios by considering 2 × 2 tables made in this game.

Another alternative that has been successful is to instruct the students to choose the entries in row 1 as numerators for fractions with the denominators selected from the numbers immediately below in row 2. Thus from the table above:

$$\frac{2}{18} = \frac{6}{54} .$$

Extend this game to 3 × 3 matrices. For example: 2, 4, 8, 16, 32, 64, 128, 256, 512 can be the consecutive entries for a 3 × 3 multiplication table. Here the constant factor was 2.

2	4	8
16	32	64
128	256	512

1. Can the marginal entries be found?
2. Are the ratios equal for adjacent row or column pairs?

Play with multiplication tables has proved to be one of the best ways to accomplish the following:

1. Consideration of the factors of a number.
2. Consideration of the common factors of two or more numbers.
3. Consideration of the greatest common factor of two or more numbers.
4. Consideration of primes.
5. Consideration of composites.
6. Consideration of divisibility.
7. Consideration of ratio.
8. Consideration of the equality of two rational numbers.

This last item, number 8, exemplifies the value of games of this type. Since many of the children note that when the diagonals have products that are equal, a set of marginal entries is possible, when they consider forming fractions as in game 5 from the entries in the table, they often conclude that a condition that $\frac{a}{b} = \frac{c}{d}$ is that $ad = bc$.

APPENDIX D

Report of the
Conference for the Development of
Mathematical Puzzle-Problems and Games

June 7-18, 1965

Stanford University

Stanford, California

INTRODUCTION. It is a common occurrence for teachers to have students who are fascinated with mathematical puzzles but who show little interest in the usual textbook exercises. Furthermore, even those students who faithfully do their problem assignments are rarely stimulated and challenged by the problems themselves. These observations suggest that the effectiveness of problem assignments would be very greatly increased if the usual drill-type exercises could be systematically replaced by puzzle-type problems and games which make use of the same mathematical skills. However, the construction of such problems and games is not easy, and most of the development has been directed toward recreational uses. Nevertheless, many educators feel that such material has a great potential for educational use, and have urged that a concentrated effort be made to develop mathematical games and puzzle-problems for this purpose. A proposal along these lines was made to the School Mathematics Study Group which has broad interests in mathematics curriculum development. After obtaining approval from the National Science Foundation, SMSG agreed to support a two-week conference to explore the possibilities for the development of mathematical games and puzzle-problems which would be appropriate for educational use. The basic aim of the conference would be to generate ideas. There would be no construction of mathematical gadgets or preparation of text material.

THE CONFERENCE. The conference was held at Stanford University during the period June 7 - June 18, 1965 with the following mathematicians participating:

Professor R. P. Dilworth, California Institute of Technology
Dr. Walter Jacobs, Department of Defense
Professor William Lister, The State University of New York
Dr. Frank Sinden, Bell Telephone Laboratories
Professor C. J. Titus, University of Michigan
Mr. Herbert Wills, University of Illinois
Mr. Robert Wirtz, Carmel, California

All of the participants have extensive interests in the area of mathematical recreation and most of them have been involved in a variety of curriculum development activities.

The formal sessions of the Conference were essentially brainstorming sessions in which a variety of ideas for puzzle-problems and games were proposed by the participants for consideration by the group as a whole. The items which survived after detailed discussion and criticism were assigned to conference participants to be written up for the final report. On the average one such session was held each day. The remainder of the time was devoted to an extensive examination of the literature on mathematical recreations; to modifying, extending, and translating known recreational items in order to make them suitable for educational use, and to preparing reports on the results of these endeavors. The principal output of the conference is the collection of working papers which is attached as an appendix to this report.

CONCLUSIONS AND RECOMMENDATIONS. First of all, it is clear that the development of an entirely new class of puzzle-problems or games is a very difficult undertaking. The abilities required are not unlike those required in developing an entirely new line of mathematical research. The essential feature which distinguishes good puzzle-problems from ordinary textbook problems is the sense of elegance, novelty, and in some cases, suprise which they generate. Likewise, an effective game must present a natural, attractive challenge to the players. But it is exactly these features which characterize first-class mathematical research. Hence, it is to be expected that outstanding contributions to the art of puzzle-problem construction will be rare. However, it also seems to be the case that the discovery of a new idea will frequently lead to a wide variety of puzzles and games. The many puzzles and games which have developed from S. Golomb's invention of polyominoes illustrates this point. During the relatively short period of the Conference, some novel ideas for puzzle-problems and games were produced. However, it will take further development and experimentation to determine how effective and how broadly applicable the ideas are. In any case, it was the unanimous feeling of the participants that a joint effort such as this is an effective way to stimulate ideas in this area. They further recommended that similar conferences of this type be held from time to time to encourage creative activity in this area and to make use of the ideas and talents of other people. It was felt that a two-week period was about the right length of time since after a fortnight of intensive effort the wells of creativity appear to temporarily run dry.

Many of the working papers do not present entirely new ideas for puzzle-problems and games but rather give ideas for the development of known material in order to increase its potential for educational use. This is clearly a very important activity for such a conference and in many cases much imagination and resourcefulness is required.

No attempt was made to present the material in polished form. The working papers serve merely to get the ideas down in a preliminary form suitable for later experimentation and modification for use in curricular material.

The members of the Conference are agreed that the next stage in the development of this material should be experimental testing on an item by item basis. This will require a good teacher as well as a mathematician. Furthermore, simply turning the material over to a good teacher will not be enough since the objectives which the inventor had in mind may very well be lost in the process of adopting the item to classroom use. Perhaps the ideal arrangement would be to have the experimentation done by a team of two -- a good teacher and a mathematician. If the mathematician involved could be the man who prepared the item, so much the better. Under any circumstances it is the consensus of the Conference that until there has been such experimentation, the material should not be widely circulated. In particular, it should not be made available to textbook writers in its present form.

After experimental testing, the next stage should be the development of some sample text material incorporating the tested items. This could be a responsibility of a writing team. On the other hand, it is the feeling of the Conference participates that this should be approached in an imaginative way. For example, rather than trying to fit puzzle-problems and game activity into the traditional text format, it might be much more effective to design the text material around the problems and games themselves, so that the learning of mathematical facts and the development of mathematics skills is a natural by-product of the interest in these activities.

Finally, the participants wish to express their appreciation for the cheerful and hearty cooperation of the SMSG staff.

R. P. Dilworth, Chairman

APPENDIX

The working papers produced by the Conference participants are collected in this appendix. It must be emphasized that these papers were prepared in order that there would be a permanent record of the ideas generated during the Conference. No attempt was made to polish the presentation nor was any effort made to be complete and detailed in the discussion of the item. Rather the motivation was to get the basic ideas down in writing as quickly as possible and to get on to the generation of more ideas. Furthermore, the form of the presentation was that which was most convenient for the particular participant involved, and no effort was made to get uniformity.

It should be noted that the working papers vary from short accounts of very particular items to broad descriptions of whole classes of games and puzzles. The grade level extends over the primary and secondary spectrum. In many cases, the activity could be profitable for primary students while an insightful analysis could only be carried out by advanced secondary students. For example, Nim can be played by primary students, but they are hardly able to analyse the game. Finally, it should be mentioned that the items fall into three categories -- classroom activities, individual puzzle-problems, and games. Some of the items naturally fall into more than one category, but a rough distribution of the items is as follows:

Classroom presentation: 1, 2, 3, 6, 11, 20, 21, 24
Puzzle-problems: 10, 14, 15, 17, 19, 21, 22, 23, 24
Games: 1, 4, 5, 6, 7, 8, 9, 12, 16, 18, 25, 26, 27

A rough classification of the items according to grade level is as follows:

Grades 1, 2, 3, 4: 1, 2, 3, 5, 7, 8, 9, 11, 12, 13, 16, 17, 18, 20, 26
Grades 5, 6, 7, 8: 1, 2, 3, 4, 6, 10, 11, 13, 15, 16, 17, 18, 19, 21, 22,
23, 24, 25, 26, 27
Grades 9, 10, 11, 12: 6, 15, 17, 19, 21, 22, 24, 27.

TABLE OF CONTENTS FOR APPENDIX D

Arithmetic Function Games 120
Addition and Multiplication Tables 122
Magic Squares and Numerical Tic-Tac-Toe 124
A Place System Game (Grades 2-4) 125
Number Games With Cards 127
Classroom Activities Based on Computer Concepts 130
Jumping Game 143
Sequence Games 146
The Game of "Y" 150
Red, White and Blue 154
Nim-Type Number Games 156
Fibonacci Problems 159
Geometric Puzzles 162
Number Theoretic Games on a Number Line (Grades 3-6) 179
Puzzles Concerning Numerical Oddities 181
Trains of Rods. 184
Polyomino Problems 186
Powers and Primes 192
A Problem in Dividing Fractions 195
Logic Puzzles 197
Does the Order Make a Difference? 202
A Game With Factors and Multiples (Grades 4-6) 203
A Stochastic Number Line Game 205
Games on Directed Graphs 207
Elementary Theory of Areas 209
Map Coloring Problems 213
A Combinatorial Problem Solved by Geometry 215

ARITHMETIC FUNCTION GAMES

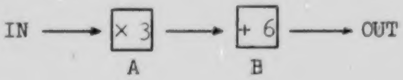
A great variety of classroom games are suggested by the small sample listed below. They should, incidentally, provide practice in basic arithmetic skills, but their focus is primarily on (1) properties of numbers with respect to operations, (2) the function idea, and (3) the problem of inferring a relationship from experimental data.

I. Linear operators (grades 2-4)

A student (the "machine") picks a linear function, e.g. multiply by 3 then add 2. The class feeds inputs to the machine which then announces the output. The problem is to identify the machine's operation.

II. Composites of operators (grades 3-5)

The teacher assigns to one student a multiplier (say 3) and to another an addend (say 6). The class does not know which is a multiplier nor what numbers have been assigned. The class hooks up the students in series and suggests inputs to A. He whispers his output to B who operates on it and announces the composite machine's output to the class.



The problem is to infer what each component is doing. The idea of equivalent machines and of standard machines, defined as those in which the multiplier is A, will come up naturally here.

III. Remainder operators (grades 4-6)

In one version a student selects a divisor (say 6). For every positive integer given to him he computes the remainder after division by 6 and announces it to the class. The problem is to identify the divisor.

A variation is to have the same (unknown) number less than say 100 put through selected known remainder machines until it is identified.

IV. A classification game (grades 3-6)

The "machine" format can be used in a number of games which amount to the identification of a rule which defines an equivalence relation. For example, a student thinks of a way of classifying all positive integers in n classes which he calls 1, 2, -- n. As integers are given to him he puts them in the appropriate one of these classes. The problem is to find an equivalent of his "rule". The same sort of game could be played with rational inputs where the classification might be according to possible denominators, properties of decimal expansions, etc.

W. Lister

ADDITION AND MULTIPLICATION TABLES

A number of tricks and puzzles can be based on the notion of a table for an operation. They should promote familiarity with general properties of the operation and with specific relationships as well. They also offer an early opportunity for students to make up simple puzzles.

An example of a complete addition table is:

	3	5
2	5	7
4	7	9

Our examples assume the children are equipped only with the natural numbers. The activities described below are based on incomplete tables; those in which the marginal entries have been omitted.

I. Introductory tricks (grade 2)

To start, the teacher could draw a blank 2 by 2 table on the board and ask the class to choose a number and box in which the number is to be written. The class then chooses a second number which the teacher writes in a box of her own choice. Next the class fills in either of the remaining boxes arbitrarily and the teacher chooses a number for the remaining box so as to make the diagonal sums equal. Thus

6	

6	4

6	4
	3

6	4
5	3

is a possible sequence. An analysis of the table should reveal

1) Why the teacher did not complete a diagonal with the 2nd entry.
2) That any table with equal diagonal sums has a constant row difference and constant column difference.
3) That any such table is an addition table with margins removed.
4) That the sum of all the entries in any such table is even.

II. Higher order tables (grades 2-3)

The "magic" property of a 3 by 3 addition table can be explored; namely the constancy of the sum of any 3 entries representing all rows and columns. The following sorts of puzzle-problems can then be given:

1) Can a table be made with equal diagonal sums which is not an incomplete addition table?
2) Which 3 by 3 tables have unique margins?
3) What tables with some blank cells can be uniquely filled in to make an addition table?
4) What is the smallest number of entries which are not part of any addition table?

III. More difficult problems (grades 4-6)

Consider higher order addition tables in which the entries are consecutive, as follows:

3	4	5
6	7	8
9	10	11

a) What magic sums can be produced by such a table?
b) Can such a table of a given order with a prescribed magic sum be made?

IV. The multiplicative analog (grades 3-5)

New features such as rational row and column ratios and various forms of their fractional representations appear in this case. The problem of filling in tables with interior blanks is more interesting and provides a source of puzzle problems with which students can challenge one another.

For example:

	6	
7		4
	9	

has a unique completion, as does

0	1	
		1
	0	

W. Lister

MAGIC SQUARES AND NUMERICAL TIC-TAC-TOE

A game can be used to introduce conventional magic squares (row, column, and diagonal sums all equal) and, more important, to exhibit the isomorphism notion. On the one side it leads to the general topic of arithmetic configurations and on the other it leads to codes, isomorphic structures, and intertranslatable systems.

To start, scramble a 3 by 3 magic square, for example by putting the entries in natural order.

15	1	11
5	9	13
7	17	3

1, 3, 5, 9, - - - - - 17

The class is shown only the sequence. The teacher and class alternately choose numbers (without replacement) from the sequence each attempting to make the sum 27 using one or more of his own choices.

T	9		11		5		
S		15		7		13	etc.

This game would probably result in a win for neither side, but at first the teacher would frequently win as first player and never lose. The strategy can be revealed by drawing an associated magic square.

With the given square no sums of 1 or 2 entries can win and with 3 entries only the row, column and diagonal sums win, hence the game is equivalent to tic-tac-toe. The question arises whether all magic squares yield games equivalent to tic-tac-toe, and whether in particular the sum of 2 entries or the sum of 3 entries which do not form a row, column or diagonal can win.

Eventually, of course, students will attempt to find a systematic method for constructing 3 by 3 magic squares.

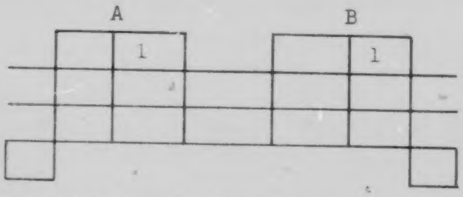
R. Wirtz

A PLACE SYSTEM GAME

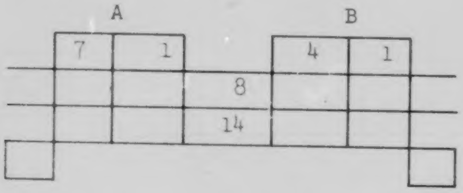
(Grades 2-4)

The game idea described below has a simple strategy which requires insight into generalized place value enumeration systems, and the play sharpens skills in counting by groups or division with remainder.

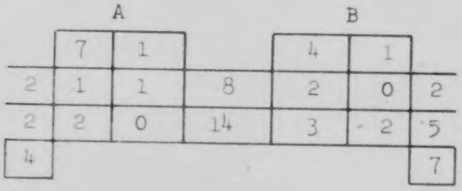
A simple version can be played on the following sheet:



The play begins with the choice of a counting base. A chooses his (7) and B his (4) entering each as shown below. A and B then each choose a number to be expressed in each base. This choice should probably be limited at first, say from 1 to 30. Suppose that A chooses 8 and B 14.



Each player now expresses each number selected as a linear combination of 1 and a base, then sums the multipliers. The player with the lower sum wins, A in this case.



The game can be expanded by requiring multiple bases, and larger or more numbers to be expressed. Here is a play of such a variation.

	A				B			
	7	3	1		8	4	1	
2	2	0	0	14	1	1	2	4
4	2	0	2	16	2	0	0	2
1	1	0	0	7	0	1	3	4
4	1	1	2	12	1	1	0	2
11								12

Here A chose 14 and 7, B 16 and 12.

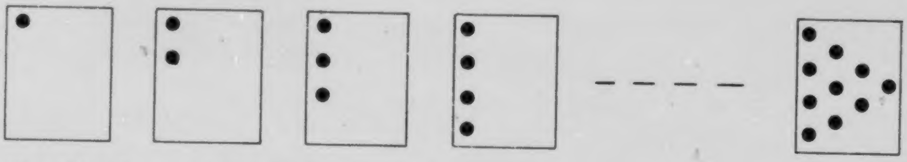
An offensive strategy is rather obvious but a defense is more dificult.

R. Wirtz

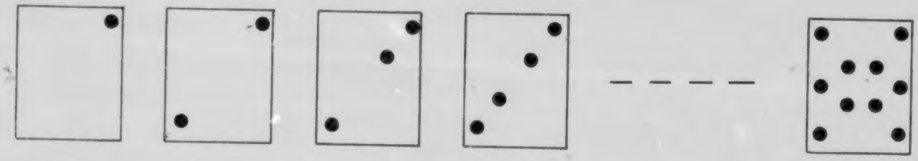
NUMBER GAMES WITH CARDS

Using a simplified card deck, or even sticks of various lengths, a number of games involving matching, ordering, counting, or adding can be adapted from conventional games for use in even the very early grades. A sample of these is described briefly below. These games have the disadvantage of poor suitability for general classroom activity but may be useful as small group activities which are continued outside of school. They provide a natural context in which basic numerical skills can develop, and their strategic aspects afford an introduction to problem analysis.

In order that card games be playable in the first school years special standard decks are needed. A first deck might consist of 30 (or 40) cards marked with 1 to 9 dots in 3 colors and configurations. For example the "red" suit might look like this:



and the "blue" suit like this



The next deck might consist of both numerals and dots, and a further version might be marked only with numerals in the various colors.

I. War (grades k-2)

Some or all of the cards in the decks described above may be used in this and the succeeding game. A main objective in the earliest stage is to develop sensitivity to equality and order with respect to number. Numerals are not at first available and not needed. Later a deck marked with numerals should facilitate number comparison and the number-numeral associations.

Cards are divided among 2 (or 3) players. Each player selects the top card of his own stack which is face down and turns it up between the players. The player with the higher card takes all those turned up and puts them aside. In case of a tie the cards are left in the center and are awarded to the player who wins the next comparison. Play continues until the cards are exhausted. The winner is the player with the most cards put aside.

II. Solitaire (grades k-2)

This is a simple exercise in number matching. The teacher places one card of each numerical valve in natural order in a row in front of the player. The remaining cards are shuffled and put face down in a stack in front of the player. He turns the top card face up and tries to match its number with one in the row, placing it on top of the one it matches. If he proceeds properly he should finish with 3 (or 4) cards in each pile in the row in front of him. The same game played with a deck marked only in numerals transfers the literal matching to the matching of symbols.

III. Fan-tan (grades 1-5)

This is a simple game involving matching and ordinality. Strategic elements are present but uncomplicated. The game operates best with 3-5 players.

With 3 players the cards are dealt around and the player holding the 5 of (say) red puts it face up in the center. Each player in turn plays a 5 as the start of a new row or plays a card next in numerical sequence to one already played and of the same color. The cards are placed so that each row corresponds to a color and so that the cards in each row are in proper order. A player who cannot play surrenders his turn, but any player who can play must do so. A player with more than one play selects one which will promote his own play more than that of his opponents. The first player to play his last card wins. Each hand can be scored by awarding the winner the number of unplayed cards or by counting against each player the number of cards he has left in his hand.

IV. Rummy (grades 2-)

This standard game adapts directly to the simplified decks. Matching, forming sequences, and the idea of defensive play are its main characteristics.

V. Cribbage (grades 2-)

This is basically a two-person game which is strategically more complex than either fan-tan or rummy, and which affords practice in matching, ordering, adding, subtracting, and forming sequences.

Although the rules of conventional cribbage are readily learned in play, they are not easy to recite concisely and will not be described here. A few simple modifications suffice to adapt cribbage to the simplified decks. The 15 sum for which 2 points are awarded can be replaced by 10; and the upper boundary of 31 can be replaced by 21. In the simplest version the "crib" can be eliminated and 4 instead of 6 cards dealt to each player. The game is best scored on a board with parallel rows of holes along which each player moves a pair of pegs to keep track of his accumulated points.

W. Lister
H. Wills

CLASSROOM ACTIVITIES BASED ON COMPUTER CONCEPTS

A. Introduction

1. The aspect of computing devices that is probably most important for the teaching of arithmetic is the way they express complex procedures by the repeated use of simpler ones.

2. This aspect can be the basis for a continuing series of classroom activities which teach both arithmetic structure and teamwork, and at the same time can stimulate and challenge the class.

3. Furthermore, such activities provide a valuable preparation for the eventual study of computer programming; this is already being taught to some high school students, and it is highly likely that it will be taught to even younger pupils.

B. An example

1. The appended "Binary counter game", which uses five students to simulate the operation of a mechanical counter, illustrates some of these points.

2. Each student acts as a "flip-flop", and changes his state (from arm up to arm down, and vice versa) each time he receives an effective signal. The signal for the units position indicates a unit to be counted by the group, and for the other positions it indicates a carry from the preceding position.

3. Each participant has a fairly simple rule governing his activities, but it will still require practice and attention to avoid mistakes. Since the game can come out properly only when everyone has done his part correctly, the game provides training in teamwork.

4. Furthermore, the game gives a concrete expression to the concept of carrying. In understanding why the mechanical rules lead to correct results, the class will obtain a better grasp of number bases.

5. The unexpected way in which the device for counting forward becomes a device for counting down, simply by reversing the rule for carrying, should serve to introduce the idea of programming and the use of basic components to produce different results by varying the rules by which they are operated.

6. The use of three-stage elements (no arms up, one arm up, two arms up) rather than two stage elements, but with the equivalent rule on carrying, is valuable in emphasizing the common structure that underlies representations that employ different number bases. The teacher can bring in the fact that the same principle, in the decimal number system, is used in such counters as the odometer in an automobile.

7. There are a limited number of elements in the counter. This will compel attention at some point to the fact that there exists a largest number that the counter can register, and counting forward from that number returns the counter to zero. Thus addition in a modular number system is introduced naturally.

C. Programming

1. The example shows that even the simplest aspect of computing device is rich in possibilities for exploring the world of numbers.

2. Slightly more advanced projects bring in the flexibility provided by the commutative, associative and distributive laws, and force attention to the details of programming in order to achieve the answer in the best possible way.

3. Programming may be approached in the most limited way, so as not to make unreasonable demands on teacher and class, and still offer substantial challenges to all concerned.

4. The second example focuses on only one aspect of programming, namely, the flexibility to rearrange the order in which commutative operations are performed. This has a significant effect on the speed that can be attained, and so can be made the basis of a competitive activity.

5. The second example, called "A race involving base 3 addition", proceeds one step beyond counting, to adding a series of numbers by accumulation. Even this moderate advance complicates the programming, the mathematical implications, and the demands for precision by the performers.

D. Further activities

1. Even the relatively easy activities of the type indicated by the two examples call for a clear understanding of the principles that are involved.

2. In addition, it is evident that such activities should be carried on through the various grades, in order that the students' understanding can grow.

3. These considerations suggest that games of the type which have been described should be conducted by special demonstration teachers, until classroom teachers have become sufficiently familiar with the principles.

4. If special teachers are available, it is possible to go into much more varied kinds of activities, that the average teacher could not reasonably be expected to cope with.

5. This is illustrated by the example "Adding a sequence of positive digits"; although arithmetically this example is no more advanced than the second one, it is carried out by means of a more complex program. The use of recursive loops and conditional transfers gives some feeling for the way digital computers are actually programmed.

6. An example such as this can provide a starting point for a class project, with the goal of extending the program to handle negative numbers.

7. Two additional examples involve the calculation of the greatest common divisor of two numbers and the determination of the first 25 primes. In contrast to the early examples which perform simple arithmetic operations with rather complex programs, Example 4 does a more advanced calculation with a very simple procedure. Example 5 introduces the idea of sub-routines at a very elementary level.

E. Using the examples

1. The discussion of examples has omitted explanations of how in detail the class carries out each activity. This has been done in part to keep the descriptions brief, and in part because some classroom experimentation is appropriate.

2. On the other hand, the teacher must clearly understand what is going on. A substantial amount of confusion is to be expected in the early stages of the activity.

Example 1. A Binary Counter Game

Objective: To demonstrate counting in the binary number system.

Participants: Six children, including the Signaller, the Ones Position, the Twos Position, the Fours Position, the Eights Position, and the Sixteens Position.

Description: Each Position during the game exhibits a digit by holding up his arm to represent "1", keeping it down to represent "0". Thus, the five Positions can together exhibit any number from 0 to 31 in binary representation. Counting is performed mechanically, each Position after the Ones Position following the same invariable rule. The procedure of the game depends on "reversing": raising the arm when it is down (changing "0" to "1"), or dropping it when it is up (changing "1" to "0"). To take care of carrying, Twos Position "reverses" whenever Ones Position drops his arm; and a corresponding rule holds for the other Positions. The children must observe a strict sequence, or confusion will reign.

Procedure: The Signaller stands in front of the class on the right side, facing left. The five Positions stand in a file facing him, in the order: Ones, Twos, Fours, Eights, Sixteens. The game proceeds in cycles, each cycle consisting of the following steps:

1. The Signaller raises his arm.

2. The Ones Position "reverses" his arm, and then makes a quarter turn to the right, so as to face the class.

3. Each other Position becomes active as soon as the one in front of him completes his right face. If the one in front has raised his arm, the active Position simply does a right face. If the one in front has dropped his arm, the active Position "reverses" his arm, and then makes his quarter turn to the right.

4. When Sixteens has finished his right face, all five Positions are facing the class. At this point, they exhibit, by the positions of their arms, a representation of a number in binary form.

5. The Signaller now claps his hands, and all five Positions do a left face, without changing the positions of their arms. They are again ready to start a new cycle.

Discussion: If all Positions initially have arms down, corresponding to the number "0", then the number represented at the end of any cycle is the number of the cycle:

	16's	8's	4's	2's	1's	
Start ---	Down	Down	Down	Down	Down	$00000_2 = 0_{10}$
After one cycle ---	Down	Down	Down	Down	Up	$00001_2 = 1_{10}$
After two cycles --	Down	Down	Down	Up	Down	$00010_2 = 2_{10}$

etc.

If initially some number other than "0" is represented, then at the end of any cycle the number represented is the sum of the initial number and the number of the cycle, provided that this sum has not exceeded 31. When the class understands the game, the students may be asked to predict what the result will be in counting 4, starting with 30.

Extensions: To count down from an initial number, the rule is that a Position (after Ones) "reverses" arms when the Position in front raises his arm, and simply does a right face when the Position in front drops his arm. It will take a bright student to guess the effect of this change in the rule.

By making use of both arms, the same type of game can demonstrate counting to the base 3. Here the digits are represented by

0 - Both arms down
1 - Right arm up
2 - Both arms up

Example 2. A Race Involving Base 3 Addition

Objective: To demonstrate addition in the base 3 number system by simulating the operation of a mechanical adder.

Participants: Two or more teams, each of five players, who take parts respectively of the Ones Position, Threes Position, Nines Position, Twenty-sevens Position, Eighty-ones Position.

Description: Each team performs as a base 3 adding machine, and attempts to add a series of six base-3 numbers as rapidly as possible. The teams may compete in pairs, or race against time. They can be given freedom to choose among different procedures, ranging from deliberate and reliable to rapid and precarious methods (precarious because of the likelihood of a mistake). A class discussion of possible methods can help to bring out the commutativety of addition at the sub-operation level.

Procedure: The method of operation follows principles similar to those described in connection with the Binary Counter Game. Each Position exhibits a digit in base 3 by holding up a corresponding number of arms, and adds another digit by counting forward from the current position. For example, if one arm is up (corresponding to the digit "1") and the digit "2" is to be added, the Position raises the second arm and silently counts "One", then drops both arms and silently counts "Two". Having reached the digit to be added, he is done. The next higher position must have been watching, and must add "1" as a carry at the point where the other has dropped both arms.

In detail, the procedure for adding a base 3 number to one already represented by the arm positions of the team is as follows; the step-by-step method is described:

1. Ones Position adds the units digit of the new number, which is written on the blackboard. He signals that he is done by making a quarter turn to face Threes Position.

2. Threes Position makes a quarter turn to face the blackboard, and adds "1" if there has been a carry signaled by Ones dropping both arms. He continues by adding the threes digit of the number on the blackboard, and when he is done he turns to face Nines.

3. This continues until Eighty-ones has finished his addition. At this point he calls out "Ready"; Ones turns to face the board; all others turn to face him; and the team is set to start adding the next number.

Discussion: To work for speed, the procedure is modified. Each Position can start to add his next digit as soon as two conditions are satisfied:

1. The Position below him has either dropped both arms, or has signalled that he is finished by making a turn.

2. The Position above him has signalled that he is ready by turning toward him.

(The first rule does not apply to Ones Position, nor the second to Eighty-ones.) The validity of these rules is not obvious, and classroom exploration of them is useful.

The team must exercise care in signalling, or a mistake will almost certainly be made. A team will have to do some practicing to achieve speed with accuracy.

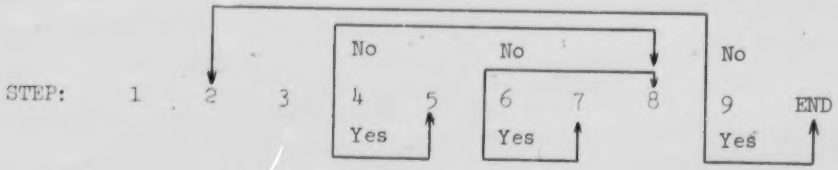
Example 3. Adding a Sequence of Positive One-Digit Numbers

Objective: To illustrate the use of recursive and conditional controls in programming.

Participants: The teacher and five students, the Counter, the Number Bank, the Units Position, the Tens Position, and the Overflow Indicator.

Description: The Number Bank holds a list of positive one-digit numbers, which he will pass to Units one at a time, in accordance with the count indicated by Counter. Units will add each digit to the one he currently holds, and will announce whether a carry is required, in that case holding only the units digit of the sum. Tens will add one when a carry takes place, and will announce whether a further carry is required, also retaining only the units digit of his sum. Overflow's only job is to register a "1" in the case that Tens has announced a carry, as indication that the total shown by Tens and Units is no longer correct. (The function of Overflow cannot be removed by replacing him with a Hundreds, because it would then be necessary to indicate when Hundreds overflows. This distinction between the finite capacity of a physical representation and the transfinite cardinality of the set of positive integers has to be grasped at some stage.)

Procedure: The teacher has the following diagram on the board:



He points to the number of each step as it is carried out, and calls out the name of the participant who is to act at that step. Also, he progresses from each step to the one with the next higher number, except at steps 4, 6, and 9, where he follows the arrow corresponding to the result of the announcement made at that step.

The activity begins with Tens, Units and Overflow all holding 0. Number holds the serially numbered list of N digits. (N may be 20, for example.) The teacher points to the 1 in the Step diagram and calls "Counter"; Counter then holds the count 1 and reads "1" to Number. The teacher points to Step 2, and calls out "Number." Number then reads to Units the digit on line 1 of his list.

The complete set of steps is as follows (these are the different possibilities, but not the actual sequence, which proceeds as in the illustration below):

STEP	PARTICIPANT	ACTION
1	Counter	Holds 1, and reads "1" to Number.
2	Number	Reads to Units the digit on line named by Counter.
3	Units	Adds to number held the digit named by by Number.
4	Units	Calls "NO" if sum is less than 10, "YES" if sum is greater than or equal to 10. Retains units digit.
5	Tens	Adds 1 to number held.
6	Tens	Calls "NO" if sum is less than 10, "YES" if sum equals 10. Retains units digit of sum.
7	Overflow	Displays 1.
8	Counter	Adds 1 to number held.
9	Counter	Calls "NO" if sum is less than N, "YES" if sum equals N (the total number of digits to be added.)

When End is reached, the final total is given as the pair of digits held by Tens and Units, provided that Overflow still shows 0. If he displays 1, the correct answer differs from that held by some multiple of 100.

Each participant may hold a card showing the step numbers at which he takes action, and the nature of the corresponding action. Even in this case, it will help to have the teacher call his name as the number of the current step is indicated.

The following shows the sequence of calls and numbers held for the first three cycles of the activity. It is easy to verify that this sequence corresponds to the program in the case that Number's list begins 8, 3, 9, ...

STEP	COUNTER	NUMBER	TENS	UNITS	OVERFLOW
Start			0	0	0
1	Calls "1"		0	0	0
2	1	Reads "8"	0	0	0
3	1		0	0	0
4	1		0	"No"	0
8	Calls "2"		0	8	0
9	"No"		0	8	0
2	2	Reads "3"	0	8	0
3	2		0	11	0
4	2		0	"Yes"	0
5	2		1	1	0
6	2		"No"	1	0
8	Calls "3"		1	1	0
9	"No"		1	1	0
2	3	Reads "9"	1	1	0
3	3		1	10	0
4	3		1	"Yes"	0
5	3		2	0	0
6	3		"No"	0	0
8	Calls "4"		2	0	0
9	"No"				

This final call is made because N is greater than 3, and so the next cycle begins with Step 2 once more. The total "20" of the first 3 digits corresponds to the pair of digits held by Tens and Units.

Extensions: The class may undertake to extend the procedure to handle the addition of numbers of several digits. A more difficult project is to provide the capability to add negative as well as positive numbers. The simplest way of doing this involves adding the complement of a negative number, and then complementing the final result when it is negative.

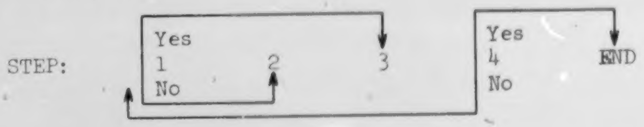
Example 4. Finding the Greatest Common Divisor

Objective: To promote understanding of the Euclidean Algorithm, and to demonstrate what can be done with very simple programs.

Participants: Three students, Control, First, and Second.

Description: Control directs the program, which involves only four steps. First and Second begin by writing on the blackboard their respective numbers, which are the pair whose greatest common divisor is to be found. The program has the effect of repeatedly subtracting the smaller number remaining from the larger, until one of the numbers is reduced to 0. The other at that point is the desired answer. The class should be expected to understand why this process will yield the greatest common divisor.

Procedure: After First and Second have their numbers on the board, Control directs the succession of steps by using the following diagram in the manner described in Example 3:



The actual steps are as follows:

STEP	PARTICIPANT	ACTION
1	First	Calls "Yes" if his number is as large as Second's number, "No" if smaller.
2	First and Second	Interchange numbers.
3	First	Subtracts Second's number from his, and retains the difference as his number.
4	First	Calls "Yes" if his number is 0, "No" if it is positive.

When the END is reached, Second's number is the desired answer.

Extensions: The class may rewrite the program so that First and Second have more symmetrical parts; in the program as given, Second has nothing much to do, and can be merged with First if desired.

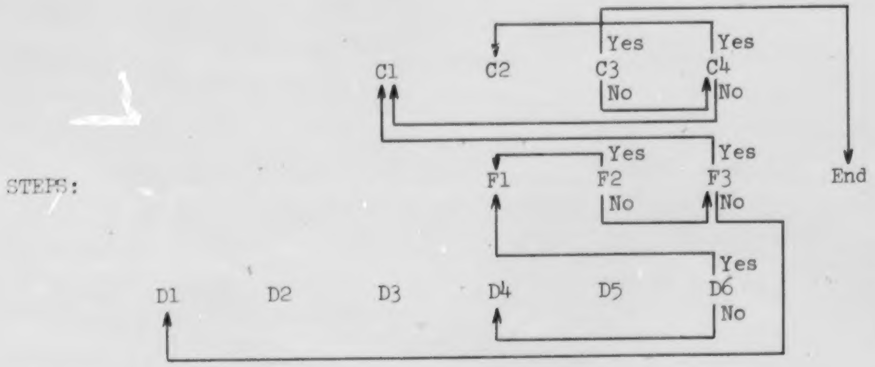
Example 5. Finding the Primes Less Than 100

Objective: To demonstrate the use of sieving to find the prime numbers, and to illustrate the use of subroutines.

Participants: Seven students, called respectively Delete, Find, Copy, List, Erase, Next, and Number.

Description: The program operates by alternating two subroutines until all primes $\leq \sqrt{N}$ are found (we take $N = 100$ for the example). Then the third subroutine tallies all remaining primes not greater than N. The participants Delete, Find and Copy control the sequence of steps in their respective subroutines, and the other four students perform specific functions. The procedure commences with the numbers 2 to N written on the blackboard, and the Delete subroutine causes the latest prime determined to be removed from this table, along with its multiples starting with the square of that prime. (The class should justify that this statement works properly.) The Find subroutine then causes the next larger prime to be located, and if this prime does not exceed $\sqrt{N}$, it returns control to Delete; otherwise control passes to Copy, which causes the remaining primes to be listed directly.

Procedure: The calculation begins by taking 2 as the first prime, and this number is held by Next. Then Delete starts his subroutine. The sequencing of the procedure corresponds to the following diagram, in which the steps of each subroutine are shown on a separate line:



These steps are described in the following table:

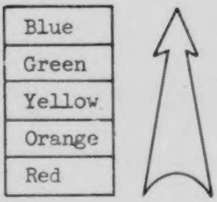
STEP	PARTICIPANT	ACTION
D1	List	Enters the number held by Next on the list of primes.
D2	Erase	Erases from the table the number held by Next.
D3	Number	Calculates the square of the number held by Next.
D4	Erase	Erases from the table the number held by Number.
D5	Number	Adds to his number the number held by Next.
D6	Number	Calls "Yes" if his sum is greater than "100," otherwise calls "No."
F1	Next	Adds 1 to the number he holds.
F2	Next	Calls "Yes" if the number he now holds has been erased, otherwise "No."
F3	Next	Calls "Yes" if the number he holds is greater than 10, otherwise "No."
C1	List	Enters the number held by Next on the list of primes.
C2	Next	Adds 2 to the number he holds.
C3	Next	Calls "Yes" if the number exceeds 100, otherwise "No."
C4	Next	Calls "Yes" if the number has been erased, otherwise "No."

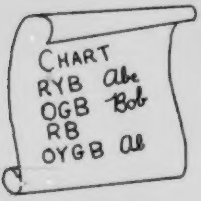
Find initiates his subroutine when "Yes" is called on Step D6. Delete resumes his when Next calls "No" on Step F3; if instead he calls "Yes," Copy begins his subroutine. At this point, all primes $\leq \sqrt{100}$ have been listed, and the numbers in the table that have not been erased are all primes. (The class should find the reason for this rule.)

It will often happen that Erase is called on to remove from the table a number that has already been erased. (When will this occur?) In this case he does nothing. It can be pointed out in discussion that erasing a blank leaves a blank, and that making use of this fact simplifies the program; otherwise the table would have to be examined before deciding to erase. Also, the need for a subroutine Copy should be discussed for its insight into the storage of information; since the numbers that remain when this subroutine begins are irregularly scattered, some method of scanning them is required. Visual scanning is so automatic that its occurrence tends to be overlooked.

W. Jacobs

JUMPING GAME





Five (the number is optional) boards 3 feet by 10 inches are joined together so that they will not separate when children jump upon them. The object of the game is to jump onto the blue (last) board successfully. A player is not successful should he land on a line or should he copy a trip made by a fellow player. A trip may consist of any number of jumps of any length as long as the jumps proceed toward the blue board. That is a player may not jump back toward the start. The trips always start on the floor and end up on the blue board. Also, a student must commit himself to a specific trip, before he starts his jumps, by recording his proposed trip on the chart. If he completes his trip successfully he enters his name along side of his proposed trip. An unsuccessful youngster must try his trip again on his next turn unless someone else has taken it successfully. If either of these situations arise he may try a new trip if he can think of one that has not been used up. For example, suppose a child commits himself to jumping directly on the yellow and then on the blue. He writes a 'YB' on the chart. Thus, all of the trips that have been used up are entered on the chart.

The player with the most successes wins the game. This game should be limited to a maximum of 4 players. The winning player starts last for the next game. Boards may be added or taken away depending upon the ability of the players. This game is useful for developing systems for exhausting all possibilities as well as giving very young students experience in extending their concepts of one, two, three, and so forth by such activities as:

If you start on the orange board and take a jump of two, where will you land?

If you started on the red board and landed on the green board, how long was your jump?

If you landed on the blue board by taking a jump of 3, where did you start?

How many jumps of 2 would you need to take to get you as far as a jump of 6 ?

If you took a jump of 2 and then a jump of 3, how far would your friend have to jump to get to where you are in one jump?

etc.

The activity also motivates the interesting problem:
How many trips are possible using, say, 7 boards?

This problem lends itself to several approaches depending upon the level of the student to whom it is presented. To illustrate a few of the possibilities we might consider approaches appropriate for elementary students, junior-high students, and high school students, respectively.

1) Suppose I had just one board. How many trips are open to me? For two boards? Three? Let's list our results:

Boards	1	2	3	4	5	...
Trips	1	2	4	8	?	...

2) Suppose I have played the game with b boards and find that I have p-possible trips. Now I add one more board. Consider each of the possible trips on the b-board game. Each of these trips ended by jumping onto the blue board, say. But now instead of jumping on the blue board, I have two choices. I can jump on the blue board or directly onto the new last board. Hence, I have twice as many possible trips as before. This gives me a recursion formula for determining the number of possible trips since I know that with 1 board I have but 1 trip.

$$\boxed{\begin{array}{c} p(1) = 1 \\ \text{and} \\ \forall b \; p(b + 1) = 2 \cdot p(b) \end{array}}$$

3) For those students who have discovered and verified that the number of subsets of an n element set is 2^n, another solution is available. Since each trip ends on the blue board, the set of boards jumped upon before landing upon the blue board determines a trip. For example, what trip corresponds to the empty set of boards? Hence, the problem simply boils down to, "How many different sets of non-blue boards are there?" This of course is the number of subsets of a set consisting of $b-1$ elements where b is the total number of boards including the blue board.

H. Wills

SEQUENCE GAMES

Many teachers are aware of the fact that children like to puzzle out the "next" term of a sequence after having been given several preceding ones. In some cases, however, the students are left with the impressions listed below.

1. There is only one sequence with the terms given.
2. There must always be a next term.
3. Sequences must be determined by some "elegant" rule.

It is unfortunate that such a pleasant pastime results in some unhealthy concepts. In order to retain these games and at the same time eliminate some common criticisims, the following suggestions are made.

Present a sequence of terms like:

$$2, 4, 6, 8, 10$$

and ask students for the next term of a sequence which has the beginning shown above. After they give their next term they are to give some kind of rule for generating the whole sequence. This approach promotes creativity and graphically illustrates that asking for the next term of an unspecified sequence is unappropriate. This activity could be started with less given terms and gradually worked up until the game gets quite challenging.

For instance one might start by giving merely two as the first term. Then students might be given:

$$2, 4$$

then:

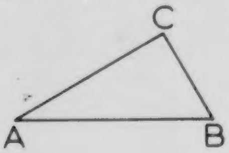
$$2, 4, 6$$

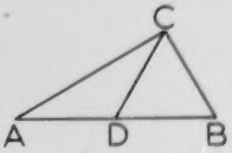
Sequences which start off:

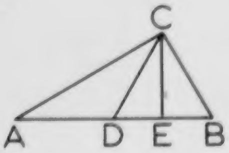
$$1, 2, 4, 8,$$

can be especially interesting since there are sequences which stem from topics in geometry which start out this way yet vary from the powers-of-two sequence at various stages. Here are a few examples:

Consider drawing line segments from the vertex of a triangle to its base. With no extra line segments drawn you have only one △ pictured. With one extra line segment drawn you have two non-overlapping triangles pictured. Now, someone who is really fast on the gun might conjecture that doubling will continue and that with two line segment we will have four non-overlapping triangles. This is not the case as is illustrated below.

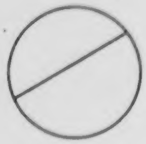


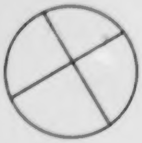




Here is a geometric situation which generates a sequence which starts out 1, 2, 4. Place chords in a circle in such a way that the maximum number of regions are formed:

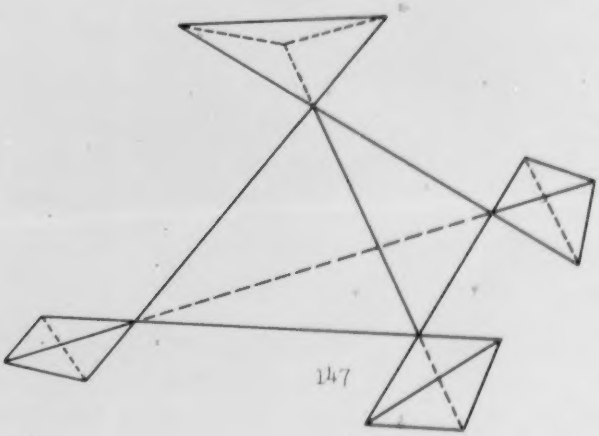
0 chords


1 chord


2 chords

In this case when we use 3 chords we get a maximum of 7 regions.

Polya frequently lectures about a geometric situation yielding a sequence which starts out 1, 2, 4, 8. The question is, how many 3-dimensional regions are formed by 4 randomly placed planes. Zero planes, one plane, two planes, and three planes result in 1, 2, 4, 8 regions respectively. He furnishes a very usefull drawing for the case of 4 planes in space:



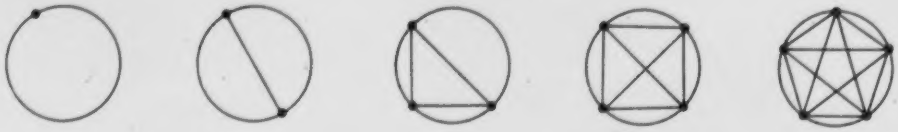
The figure shows that we have 4 regions each touching the closed tetrahedron at only a vertex, 6 touching just an edge, 4 touching each complete face and the interior of the tetrahedron itself. Hence, the number of regions is 15. It is very appealing to find a nice way for continuing the sequence suggested by this problem. So far we have:

1, 2, 4, 8, 15,

Another geometric situation which generates these terms of a sequence is described below:

1, 2, 4, 8, 16 .

Consider the number of regions within a circle determined by joining all possible chords to n points on the circle.



Number of points	1	2	3	4	5	6
Number of regions	1	2	4	8	16	?

The answer for 6 points may be either 31 or 30 depending upon the placement of the points.

Of course even though geometrical situations generate sequences students should not think that each sequence must have a geometric interpretation. Indeed, if a student does not suggest a random sequence the teacher should.

Below are some examples of starts of sequences for which students might be asked to find a next term. Once again we stress to students that their answers are based on a sequence which they have in mind and not on a keyed response.

2, 4, 6, 8, 10, ?
1, 3, 5, 7, 9, ?
1, 2, 4, 8, 16, ?
1, 4, 9, 16, 25, ?
1, 2, 3, 5, 8, ?

2, 6, 12, 20, 30, ?
1, 3, 6, 10, 15, ?
1, 9, 36, 100, 225, ?
2, 8, 18, 32, 50, ?
0, 3, 8, 15, 24, ?
2, 3, 5, 7, 11, ?
$\frac{1}{2}, \frac{3}{4}, \frac{7}{6}, \frac{2}{1}, \frac{9}{2}, ?$

The last one has an interesting possibility. If the numbers listed are identified differently like this:

$$\frac{5}{10}, \frac{6}{8}, \frac{7}{6}, \frac{8}{4}, \frac{9}{2}, ?$$

Students also enjoy constructing sequences from recursive rules which yield short cuts. There are several examples of these in the SMSG-NLSMA "Gold Mine".

Practice in sequence games may be used to familiarize students with certain patterns which come up again and again in mathematics. For instance we have already indicated a number of situations where the powers-of-two pattern comes up in our discussions on the Jumping Game and the Trains of Rods. A sequence game may be appropriately brought into classroom activity when it precedes a topic involving the same or related pattern. They may also be used immediately following the discovery of a pattern so as to give students more experience with it. This is important for kids since they could not easily discover a pattern with which they were unfamiliar. To make this point clearer, consider a class of students which had much practice in squaring numbers. When these students are faced with such numbers as 64 and 25 they immediately recognize them as squares. Consider another class without this experience. Now face them both with questions such as:

What is the sum of the first 3 odd numbers?
What is the sum of the first 5 odd numbers?
What is the sum of the first 2 odd numbers? etc.

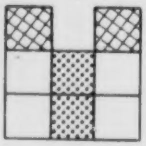
After several of these questions the class with the experience described above could easily make a discovery and apply it to answering a question such as the following while the other class would be much less successful.

What is the sum of the first 60 odd numbers?

Then under an obvious rule there would be no next term.

THE GAME OF "Y"

An investigation of associativity with other benefits including motivation for use of grouping symbols and autonomously generated drill.



The game is called "Y" because of the arrangement of the shaded cells. The cells on the arms of a "Y" go together as do those on the stem of a "Y". By "go together" we mean that whenever a numeral is entered in one of the cells across the top, a copy of that numeral must be entered in the remaining cell in the top row. Similarly, when a numeral is written in either of the cells down the middle, a copy of this numeral must be placed in the other cell.

Children may be given a start like that shown below and asked to complete the Y.

12		
	3	

They should eventually become quick at this. So that in this particular case they would have:

12		12
	3	
	3	

The shading has been left off here since in practice we have used colored overlays. Once the students acquire familiarity with the game the shading or color is not needed. Now, the teacher may insert a new numeral and designate an operation say, subtraction. The board might look like this:

12	−	12
7	3	
	3	

The students are then asked to fill in the remaining cells.

12	–	12	
7	3	4	
5	3		→ 2
		↓ 8	

This of course is not satisfying to students since they would like the final cell filled with a single appropriate numeral. So, the teacher asks the class to come up with numbers to be entered on the Y and in the middle of the 1st column so that the last cell may be entered appropriately. This is neither a trivial task nor an overwhelming one and kids generate for themselves lots of drill in subtraction. They also are given an opportunity at problem solving as well as given practice in an activity which is close to substituting for numerical variables since the rules of the game specify which cells (variables) must contain copies of the same numeral; yet they do not restrict the frequency with which a numeral may be used. In fact a child may, and they often do, choose the same numeral to be copied in each of the starting cells. Many think they have solved the problem with a start similar to that shown below.

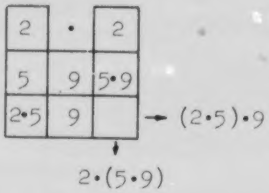
4	–	4
4	4	
	4	

Their own check reveals this to be unsatisfactory.

Once students have successfully solved the problem with subtraction they may be asked to do the same for division. Students should be cautioned not to give the game away to others since this is a fine opportunity for individual investigation.

Now that students have found that there is something about this "Y" game which is nontrivial, ask them to continue their investigation with addition. After much success with this operation ask them to find a triple of numbers for which addition will not fill the last cell appropriately. This again encourages much computation. Students should be encouraged to try fractions. Shortly they will convince themselves that there aren't triples of numbers for which the chart won't work. Then continue with multiplication.

Grouping symbols may now be introduced as a device to record the history of the 2 possible outcomes. Thus, a class may investigate a chart like this:



and since they had already concluded that the chart works for any triple of numbers they may conclude that $(2\cdot 5)\cdot 9 = 2\cdot(5\cdot 9)$ which is an instance of the associative principle for multiplication. This game points out to students that associativity is not trivial as compared to an approach of putting sticks together or merely exhibiting instances. This device emphasizes what's going on with associativity and points up the differences expressed by the left member and the right member of an instance of associativity. It also starts out with operations which are not associative and hence makes the fact that some operations are associative interesting to students. This same device may be used to check associativity of other operations such as:

$*$, where $x * y = y$

Maxing, where the maximum of a pair of numbers is the number which is not less than the other.

Taking the arithmetic mean of a pair of numbers

Taking the geometric mean of a pair of numbers

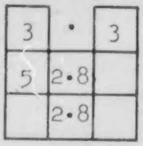
Exponentiation

After determining that finding the arithmetic (geometric) mean of a pair of numbers is not associative, it is worthwhile to investigate simple rules that will indicate how one might enter the chart so that the last cell could be entered appropriately.

After learning about associativity, several students have difficulty in seeing how associativity justifies sentences like:

$$(3\cdot 5)\cdot(2\cdot 8) = 3\cdot[5\cdot(2\cdot 8)]$$

The "Y" device might be of use here by inserting more complicated expressions in some of the cells, in the case above:



We have found that after going through experiences such as those outlined here and similar experiences for commutativity students ask questions like

Are there operations which are associative but not commutative? (*, as defined above is an example.)

Are there operations which are commutative but not associative? (Taking the arithmetic mean of a pair of numbers is an example.)

How can you decide whether an operation on a finite set is associative? (This, after having been asked a similar question about commutativity with respect to a multiplication table.)

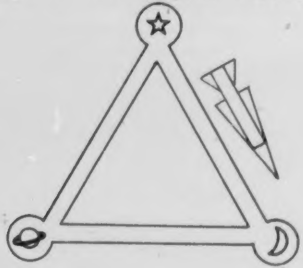
How many associative operations are there on an n element set?, etc.

H. Wills

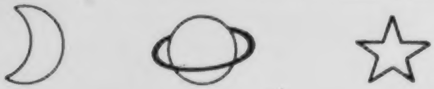
RED, WHITE, AND BLUE

A game for introducing and furnishing experience in modular arithmetic.

The playing board designates 3 paths and 3 positions. The positions are Star, Moon, and Saturn. Look at the diagram below:



Each of the two players have a marker of different colors. For instance, one may have a white pawn and the other may have a black pawn. Red and black checkers may also be used. Moves are made in a clockwise direction which is indicated on the playing board by the picture of a rocket. The reason for using such symbols as:



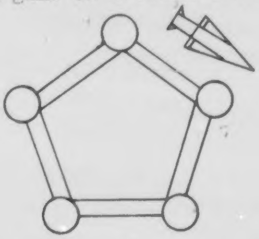
is that they carry no common connotation for order. Numerals and letters do. We have been especially careful not to label the 3 positions with numerals since we want children to think of the numbers in this game as measuring the length of trips in the clockwise direction and not as positions on the playing board. Three throwing devices determine the moves. One is red, one is white, and the third is blue. Each is a triangular prism with rounded ends whose right cross-section is an equilateral △ . Each lateral edge is considerably longer than the edge of a cross section. This shape of the throwing device guarantees that it will fall on one of the 3 lateral faces with practically equal probability.

The first player places his marker on any of the 3 positions on the board. The second player must then place his marker on the same position. The first player rolls the red device which has the symbols:

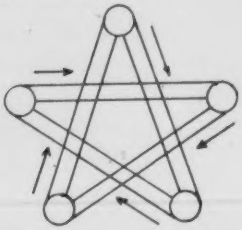
0 , 1 , 2

marked on its respective faces, as do the other devices. He moves his marker to the number of positions indicated on the down face. He then repeats this process with the white device. Now, the second player has the objective of "bumping" the first player by throwing only the blue device. His success or failure ends that half of the round and on the next round he becomes the first player. The first player to achieve 3 bumps at the end of any number of complete rounds wins the game.

This game can be adapted to 4, 5, 6, etx. position games by bringing in new throwing devices like a right prism with a square cross section or a tetrahedron etc. Initially the positions should be reached in cyclic order but later the moves should be made in such a way as to force the second player to determine abstractly the number he must throw to bump the first. For instance, after having played a 5 position game on a board like this:



change the board to one like this:



After considerable playing experience on these games, the throwing devices and board can be removed and the teacher can pose such questions as:

> Suppose you and your friend are playing on a 7 position board and your friend starts out by rolling a 4 and a 6. What must you roll to bump your friend?

Questions like this can be shortened and expressed by:

$$(4,6) \xrightarrow{\text{seven}} ?$$

During the game the teacher should bring out some of the important aspects of the game by asking such questions as:

> What 2 tosses can the first player make to make him completely safe from the second player's bump?

H. Wills

NIM-TYPE NUMBER GAMES

The typical game of this type is the game of "21" (no relation to the Las Vegas variety). This game is played by two players. The players play alternately. A play consists of choosing a number from **1-5** and adding it to the sum previously obtained. The first player to make the sum reach 21 or higher wins. At the lower grade levels this game affords considerable practice in elementary mental arithmetic. At the upper levels it provides excellent opportunities for logical analysis and problem solving.

Perhaps the simplest form of the game is to choose from the pair of numbers 1, 2 and to set the limit number reasonably small, say 8. Even very young students can play this game and may be able to find a winning strategy. In this case, the winning strategy is to first choose the number 2 and in succeeding plays to choose the number which the opponent did not select in his preceding play.

The game of "21" gives students with some problem solving ability an excellent opportunity to use logical analysis to discover the strategy of the game. The key point is the observation that one should not concentrate on the beginning of the game, but rather on the end of the game. This illustrates an important technique in problem solving, namely, if a direct attack on the problem does not work, try an indirect attack, or start from a hypothetical solution. In the game of "21", it is not hard to discover that if a player produces a sum of 15 then he is sure to win the game, for regardless of the number k which the other player selects, the choice of $6 - k$ brings the sum to 21 and wins the game. But then it is clear that the same situation holds for a sum of $15 - 6 = 9$, since a sum of 15 can then always be produced in two plays. Finally, it is clear that the proper first move is $9 - 6 = 3$. Note that the winning first move is the remainder on dividing 21 by 6 . Furthermore the winning strategy is at each play to choose the number $6 - k$ where k is the number previously chosen by the other player.

If the rules of the game are changed so that the first player to reach 21 loses, then the proper first choice is 2 and the winning strategy is the same as before.

The game can be modified in many ways. As an example of a game of this type in which the winning strategy is not unique, consider the following: Each play consists of choosing one of the numbers 2, 3, 4, 5 and the player who first makes the sum 23 or larger wins. In this game the values of the sum which lead to a win are 17, 16, 10, 9, 3, 2 . Hence the first player can guarantee a win by starting with either 2 or 3 on the first play. Furthermore, he frequently has possibilities for alternate choices during the course of the game. Note that if a student has analysed the strategy for the game of "21", it still requires some further analysis to discover the winning strategy for this modified game. If the set of numbers from which the choice is made is widely scattered, the analysis of the game may be quite difficult indeed.

A further modification of the game which introduces some geometric aspects is to play the game of "21" with sets of rectangular rods having lengths 1", 2", 3", 4", and 5". In the first play a rod is selected and placed along a line 21" long so that the left ends coincide. In succeeding plays the rods are placed along the line each next to the preceding one until the line is covered. Note that if the game is presented to the student in this form, it may not be obvious to him that the understanding of the game essentially involves a number problem. Furthermore, there is no need to have the rods or the line of integral lengths, in which case successful play may depend largely on a geometric feeling for lengths and combinations of lengths. Children in the lowest grades should be able to play this form of the game with interest.

A modification which introduces some new features is to replace the limit number by a "target" number with the aim of making the sum close to the target number. It is easy to see that if the set of numbers from which the choice is made includes 1 , then the game is equivalent to the previous form, since one can always make the sum exactly equal to the limit number. However, if 1 is not in the set, then the game is quite different. For example, let the choice set consist of the two numbers 5 and 9 . Let the target number be 53 . Then it is clear that a player wins if he can make the sum equal to one of the numbers 51, 52, 53, 54, 55, since the opponent's sums will then always differ from 53 by more than 2 . It is not hard to discover that the winning first move is 9 and the winning strategy is similar for that of the "21" game with the choice set $\{1,2\}$. In order to avoid ties, the smallest number in the choice set should be odd.

A further variation consists in fixing a given interval and a "target" (which may be either a given number in the interval or a subinterval). The moves consist of either additions or subtractions of numbers from the choice set with the restriction that the sum remains in the interval and no number can be reached twice in the course of a game. For example, the interval may consist of all numbers from 20 to 60, the "target" set consists of the numbers 37, 38, 39 and the choice set is 5, 9, 14. A game might proceed as follows.

$$35 \longrightarrow 40 \longrightarrow 45 \longrightarrow 36 \longrightarrow 50 \longrightarrow 41 \longrightarrow 27 \longrightarrow 22 \longrightarrow 21 \longrightarrow 26 \longrightarrow 21$$

The only admissible move is to 30 which loses. If a number is reached from which there are no admissible moves, the game is a draw.

The variations described so far all involve operations on integers. However, the game can be easily modified so that operations on rational numbers are involved. For example, the choice set may consist of the rational numbers $\{\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\}$ and the first player to make the sum at least 1 wins. The winning first moves in this case are either $\frac{1}{5}$ or $\frac{1}{6}$.

It should be emphasized that the numbers used in the examples are purely illustrative. A great variety of games can be formed by varying the choice sets and/or the limit or target numbers.

R.P. Dilworth
W. Lister

FIBONACCI PROBLEMS

The Fibonacci sequence of whole numbers affords some excellent problems concerning rational numbers, modular arithmetic, and inductive reasoning.

This sequence may be introduced in the following way: Consider the sequence of rational numbers constructed as follows:

$$1, \quad 1+\frac{1}{1}, \quad 1+\frac{1}{1+\frac{1}{1}}, \quad 1+\frac{1}{1+\frac{1}{1+1}}, \quad 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}},$$

Any student familiar with operations on rational numbers can reduce this sequence to the following:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots.$$

Furthermore, he should have no difficulty in coming to the conclusion that the denominator of each fraction in the sequence is the numerator of the preceding fraction and that the numerator is the sum of the two preceding numerators. The sequence of denominators

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots.$$

is the Fibonacci sequence. The first two terms are 1, 1 and each term which follows is the sum of the two preceding terms.

If we take the remainders when the members of the sequence are divided by 2 we get the sequence

$$1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$$

The student should be able to recognize that this sequence can be generated in the same way as the Fibonacci sequence but using addition mod 2. He should also note that the sequence consists of repeating blocks, that is, it is periodic.

He can then be asked to explore the periodicity for the sequence taken modulo 3, 4, 5, 6, 7, etc. He will find the periodic parts are

1, 1, 0 mod 2
1, 1, 2, 0 mod 3
1, 1, 2, 3, 1, 0 mod 4
1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0 mod 5
1, 1, 2, 3, 5, 2, 1, 3, 4, 1, 5, 0, 5, 5, 4, 3, 1, 4, 5, 3, 2, 5, 1, 0 mod 6
1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0 mod 7

A more advanced student might be able to give an argument to show that the sequence is always periodic. Namely, there are only m^2 different consecutive pairs modulo m. Thus when a given consecutive pair occurs a second time, then the immediately preceding consecutive pair also occurs a second time. Hence by repeating the argument several times, it follows that the beginning sequence 1, 1 occurs a second time. But then the portion of the sequence beginning with 1, 1 and stopping just before the second occurance is the periodic part of the sequence.

As a somewhat deeper induction, the student might discover that the number of zeros in the periodic part of the sequence taken modulo m is always 1, 2, or 4. The proof of this fact is a fairly deep result.

Returning to the Fibonacci sequence itself, the student can be asked to examine the 2nd, 4th, 6th, 8th, ... terms of the sequence, then the 3rd, 6th, 9th, 12th, ... etc. As a further exercise in inductive mathematics he may be able to discover that the mth term of the sequence always divides the nth term of the sequence if m divides n. A very bright secondary student might even be able to give a proof of this fact.

After becoming familiar with the Fibonacci sequence the student may be interested in investigating other sequences in which each term after the 2nd is the sum of the two preceding terms. The following sequence is an example:

3, 1, 4, 5, 9, 14, 23, 37, 60, 97, ...

He may be able to discover that if all of the terms of such a sequence are multiplied by the same fixed integer, then a new sequence with this property is obtained. Furthermore he can find that if two such sequences are added term by term, the resulting sequence is again of the same type. Advanced students could be asked to show that every such sequence can be obtained from the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

and the displaced Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, \ldots$$

by multiplication with constants and adding or subtracting the resulting sequences. For example, if the Fibonacci sequence is multiplied by 3 and the displaced Fibonacci sequence multiplied by 2 is subtracted from it, we get the sequence mentioned above. It should also be noted that the displaced Fibonacci sequence can be obtained by taking consecutive differences in the Fibonacci sequence.

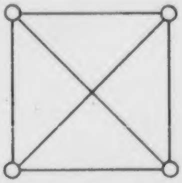
R. P. Dilworth

GEOMETRIC PUZZLES

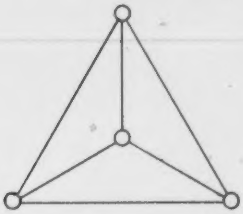
Geometric puzzles can be useful in developing spatial intuition. The questions in the sets that follow are not all puzzles, but some of them should lead to surprises and therefore have a puzzle-like quality. The first sets on penny-packing, dual patterns and plane-tiling are intended for the fourth or fifth grade level, the later ones on three-dimensional packing for the high school level.

Spatial intuition, of course, is treacherous and puzzles can be especially useful in teaching caution. Sometimes vivid mental pictures can be a positive hindrance, as in this example, which might serve as a paper and pencil exercise for primary children or a mental exercise for older children.

Join four points by straight line segments so that every pair is connected. (This takes six segments.) Can you choose the points so that no two of the segments cross? Most people visualize this:

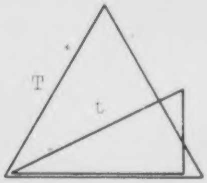


and because this picture is so clear they often fail to see the solution:

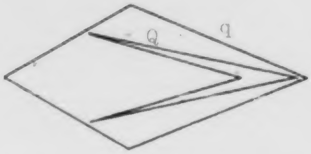


One can also ask the other question: Is it possible to choose the points so that there is more than one crossing? It is required, of course, that the segments be straight, and that no three points be collinear.

Another good problem for teaching caution is this one: Consider two triangles, a big one T and a small one t, such that every side of T is longer than any side of t. Prove that t can fit inside of T. Though plausible, the assertion is not true as shown by this example:



What about the more modest assertion: the big triangle T cannot fit inside the small tringle t. This, at least, is true. Though modest, this is a definite result and one can go on to the quadrilateral case: If every side of quadrilateral Q is longer than any side of quadrilateral q, then Q cannot fit inside q. This, unfortunately, is false:



Such dead ends should be instructive.

The following puzzle is said to have been proposed by a Russian mathematician as a test of spatial intuition. If a plane passes through the center of a cube and is perpendicular to one of the cube's main diagonals, in what figure do the cube and plane intersect? Few people can form a sufficiently vivid mental image to just see the answer, but those with experience in geometry can often recognize nontheless that the answer must be a regular hexagon. They arrive at this by observing the symmetry of the figure. Three of the cube's faces have a common vertex at one end of the diagonal and the other three have a common vertex at the other end of the diagonal. This much one can see by visualization. Since the plane bisects the diagonal and is perpendicular to it, it must, by symmetry, cut the six faces in line segments of equal length. Also, by symmetry, the line segments must meet in equal angles all around. This is not a proof of course, but it could be the basis of a proof if one wished to construct one.

Symmetry is a main theme of the exercises that follow, and could I think, be one of the main themes of all elementary school intuitive geometry.

Penny-packing

1. Put a penny on the table. How many more pennies can you pack around this one so that each one touches it? (All the pennies must lie flat on the table.)

2. Continue packing pennies around the ones you have. Can you do this so that each penny touches as many as the first one does? Do you think you could keep going indefinitely?

3. Look at the open spaces between pennies. How many pennies surround each open space? This pattern is called the "close packing of discs."

4. Can you make a pattern in which each penny touches exactly four others? Indefinite extension? Number of pennies around each open space? Can you make it so the 4 neighbors are always evenly spaced in angle? The pattern is called a regular packing if each penny has the same number of neighbors evenly spaced around it.

5. Can you make a pattern in which each penny touches exactly three others? How many around each open space? Make it regular? Compare this with the packing in which each penny touches six others. What do you notice?

6. Have made patterns in which each penny touches 3, 4 or 6 others. We will call these the 3-packing, 4-packing and 6-packing. Can you make a 1-packing and a 2-packing? This takes care of numbers 1, 2, 3, 4, 6. Can you do for more than 6? Five is missing. Can you make pattern in which each penny touches exactly five others?

7. Can you make a regular 5-packing? How sure are you

   In making up your mind, consider these questions: (a) In a regular packing does every open space have to be surrounded by the same number of pennies? (b) Could you make a 5-packing with open spaces surrounded by 3? by 4? by 5? Less than 3, more than 5?

Dual Patterns

8. Make a picture of a 6-packing. (Use a penny to draw the circles, or better a piece of cardboard with a circular hole.) Mark the centers of the circles. Now join each center to the centers of the neighboring circles by straight lines. The straight lines you have drawn divide the plane into cells. How many sides does each cell have?

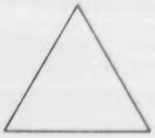
9. With a colored pencil mark the centers of the spaces between circles. Join each of these centers to the centers of neighboring spaces by a colored line. (Two spaces are neighboring if you can get from one to the other by squeezing between a pair of circles). The colored lines also divide the plane into cells. How many sides does each colored cell have? Look at the patterns of black and colored lines. Notice how the two patterns interlace.

10. On a new sheet of paper do the same thing starting with the regular 3-packing. (Include at least 30 circles) Join the centers of the circles by black lines and the centers of the spaces by colored lines. Do the patterns of black and colored lines look familiar? Compare them with the patterns you made before. What do you notice? The pattern of black lines (ignore the circles) and the pattern of colored lines are called dual patterns.

11. Imagine the pattern you would get if you started with the regular 4-packing. Can you see it in your mind's eye? Draw the dual patterns (black and colored) free-hand. Don't bother with the circles.

Tiling the plane

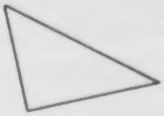
12. An equilateral triangle is one whose three sides all have the same length like this one:



Suppose you had a whole stack of equilateral triangles all the same size made of tile. Could you tile a floor with them so as to leave no spaces? (Don't worry about the edges of the floor.) What makes you think so?

13. Draw a picture of the floor pattern, but do it by drawing long lines with a ruler in three directions. (To find the directions draw one small equilateral triangle as well as you can by eye.) What happens to the triangles if you change the three directions slightly?

14. Suppose you had a stack of tiles all of this size and shape? Could you tile the plane?

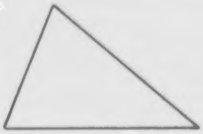


This size and shape?

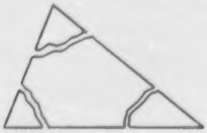
Or this?

Triangles of any size and shape, so long as they are all alike?

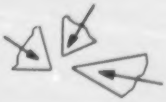
15. Suppose you had a paper triangle (like this):



and suppose you tore the corners off (like this):

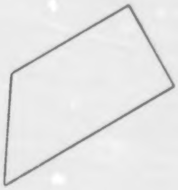


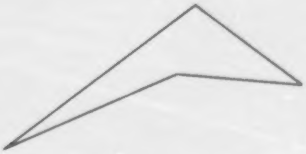
and then suppose you pushed the torn corners together (like this):



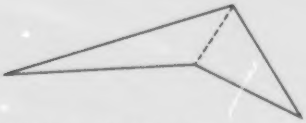
You would observe something interesting. Now, without actually doing it, what would you observe? Hint: Look back at the pictures of floor tilings that you made. Convince me that what you found out is true for every triangle.

16. Now try the corner-tearing business with various paper quadrilaterals (four-sided figures) like these:

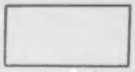




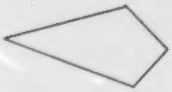
What happens? Assuming I am convinced about the triangles, convince me about the quadrilaterals. How do I know it works for all quadrilaterals? Hint: Draw a line across the quadrilateral like this:



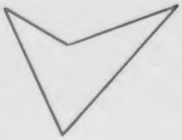
17. Can you tile the plane with these?



with these?

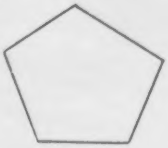


with these?



with quadrilaterals of any shape? If you think so, convince me.

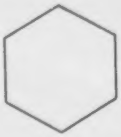
18. Here is a regular pentagon.



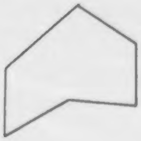
Can you tile the plane with these? (Recall your experience with pennies).

19. Are there any pentagons (five-sided figures) you can tile the plane with? Convince me.

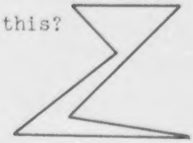
20. Can you tile the plane with regular hexagons?



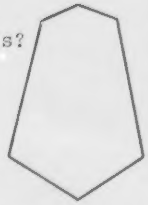
With hexagons like this?



Like this?



Like this?



With hexagons of any shape?

21. Invent a rule that tells when a hexagon will work and when it won't.

Regular Polyhedra

A polygon is regular if all its sides are the same length and all its angles are equal. A polyhedron (solid with flat faces) is regular if all its faces are identical regular polygons and all its vertices are common to the same number of faces.

1. In the plane how many equilateral triangles can you pack around a point?
2. Imagine just three equilateral triangles around a point (i.e. each with a vertex at the point) and imagine lifting them up, keeping the common vertex fixed, to form a flat-sided cone. What is the shape of the opening at the top?
3. Suppose you filled in the opening with a new face. Would the resulting polyhedron be regular? How many faces does it have?
4. Now imagine making a cone with four equilateral triangles. Shape of the opening? Can you complete a polyhedron with more equilateral triangles? Is it regular? How many faces?

*5. Now try it with five equilateral triangles. This is a little harder, but try to do it without drawing any figures or making any models. Use a combination of visualization and reasoning. Can you continue the five-sided cone to make a regular polyhedron? How many faces?

6. Six equilateral triangles?
7. Squares, pentagons, hexagons?
8. How many regular polyhedra are there? How many faces does each one have? How many vertices?
9. Interesting relationships between numbers of faces and numbers of vertices?
10. Imagine a regular tetrahedron. Imagine a point exactly in the center of each face. Imagine a polyhedron having these points as vertices. How many vertices, how many faces does this polyhedron have? Is it regular?
11. Try this for all of the regular polyhedra. What do you observe?

*12. You can fill up space indefinitely with cubes without leaving any unfilled cavities. Can you do it with tetrahedra?

*13. Describe a polyhedron all of whose faces are identical equilateral triangles but which is not regular.

Sphere Packing

*1. Imagine a stack of cannon-balls built as follows: The bottom layer has 16 balls in a 4 × 4 array. The second layer, lying symmetrically on top of the first, has 9 balls in a 3 × 3 array. The third layer has 4 balls and the fourth has one. How many balls in this stack are interior, i.e. do not touch the ground and are not on the outside?

*2. Visualize an interior ball. How many other balls does it touch?

*3. Visualize the points at which this ball touches other balls. Are these points the vertices of a regular polyhedron? How do you know? (Again, use a combination of visualization and reasoning.)

4. Now imagine a stack of cannon-balls in which each layer is a close-packed triangular array. How many balls must the stack contain to have at least one interior ball?

5. How many other balls does an interior ball touch?

6. Are the points of contact vertices of a regular polyhedron?

7. Suppose you had two cannon-balls, one from the interior of a triangular-based stack, one from the interior of a square-based stack, each with the points of contact marked. Could you tell the balls apart?

8. Imagine two close-packed layers of spheres, of infinite extent. Assuming the bottom layer fixed, in how many distinguishable ways can the top layer be put on it so that each sphere of one layer touches three spheres of the other?

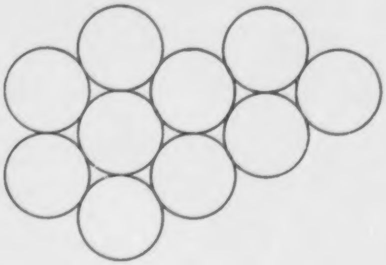
9. If the two layers are open-packed (each sphere touching four other balls in its own) and are placed so that each sphere touches four spheres in the other layer?

ANSWERS

PENNY-PACKING

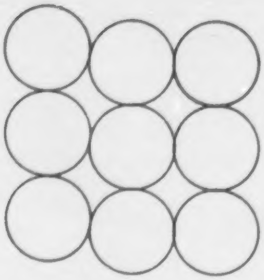
1. Six

2. Yes. Indefinite extension is possible:

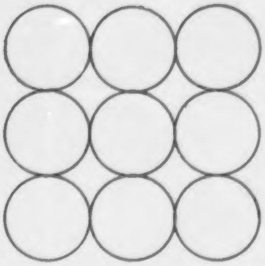


3. Three

4. Yes.

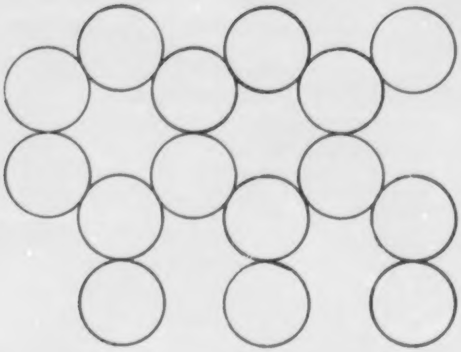


not regular



regular

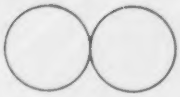
5. Yes



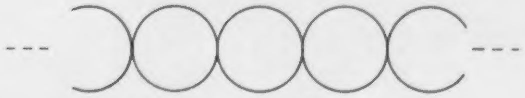
6-packing: each penny touches 6 others; each space surrounded by 3.

3-packing: each penny touches 3 others; each space surrounded by 6.

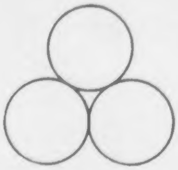
6. Regular 1-packing:



Regular 2-packing:

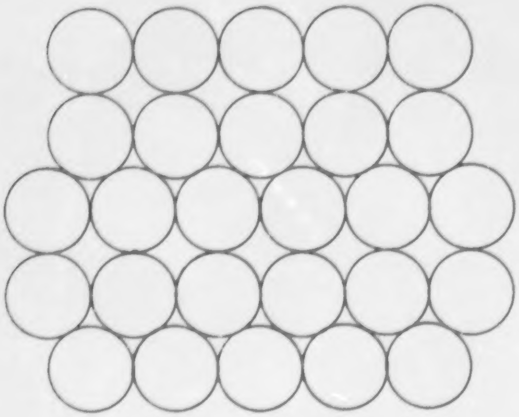


The regular 1-packing is finite; the regular 2-packing is infinite in one direction; the regular 3-packing is infinite in all directions. Here is a (non-regular) finite 2-packing:



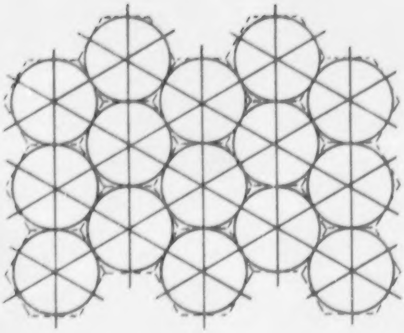
Can you find a finite 3-packing?

Each penny touches five others (non-regular 5-packing):



7. No.

8, 9.

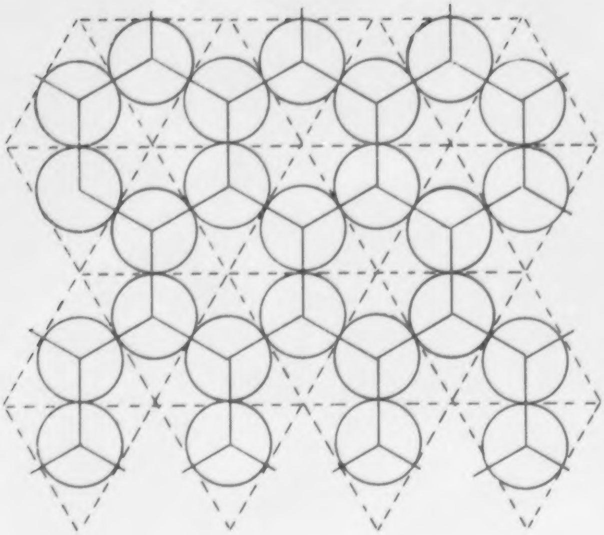


Solid lines join centers of circles; dashed lines join centers of spaces. (Dashed lines represent the colored lines)

Cells formed by solid lines have 3 sides.

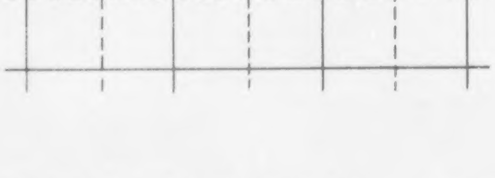
Cells formed by dashed lines have 6 sides.

10.



Aside from scale, the patterns of solid and dashed lines are the same as before with poles reversed. The dashed lines form triangular cells; the solid lines form hexagonal cells.

11.

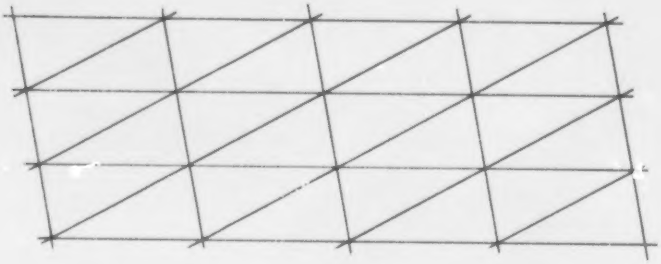


Both the solid and dashed lines form square cells.

These patterns are self-dual. The triangular and hexagonal patterns are dual to each other.

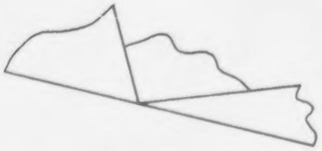
12. The tiling pattern is that formed by the solid lines in 8, or the dashed lines in 10.

13. Tiling with non-equilateral triangles:

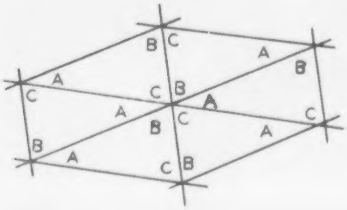


14. All questions: yes.

15. The three angles combine to form a straight angle (180°):



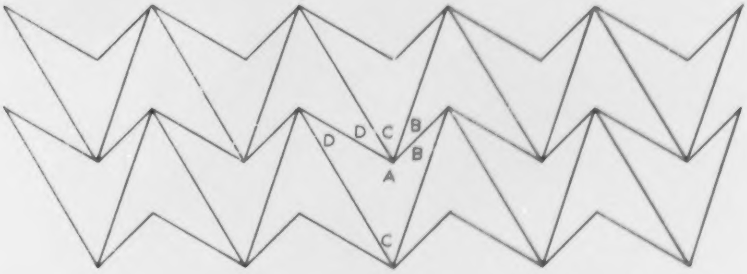
In the floor tiling note that each of the three angles (call them A, B, and C) is represented at each intersection point exactly twice:



If 2A + 2B + 2C exactly fills a full angle (360°), then A + B + C must exactly fill half of this or a straight angle (180°).

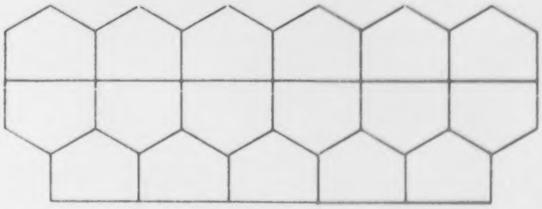
16. The four angles of a quadrilateral exactly fill a full angle (360°). Divide the quadrilateral into two triangles. The angles of each triangle fill a straight angle, hence the angles of both triangles together fill a full angle.

17. The plane can be tiled with quadrilaterals of any shape. Each of the four angles is represented at each intersection point. Example:



18. No.

19. Yes:



20, 21. I do not know necessary and sufficient conditions that a hexagon be capable of tiling the plane. A sufficient condition is that it have a pair of opposite sides that are parallel and equal in length. The regular hexagon and the first two odd-shaped ones given in 20 satisfy this condition (hence can tile the plane) but the last one does not. The last one apparently does not tile the plane.

ANSWERS

REGULAR POLYHEDRA

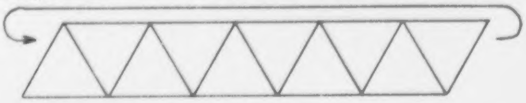
1. Six.

2. An equilateral triangle.

3. Yes. Four faces. All are equilateral triangles, and at each vertex the same number (three) come together.

4. Square opening. Figure can be completed with four more equilateral triangles forming cone identical to first cone. Each vertex is common to four faces, hence figure is regular. (Octahedron, 8 faces)

5. If the polyhedron is to be regular it must have five faces at each vertex. Three more faces are needed at each vertex on rim of cone. Form ring of 10 triangles out of strip like this:



Attach ring to rim of cone. Surmount with 5-face cone identical to first cone. Polyhedron has twenty faces (icosahedron)

6. Six equilateral triangles lie in plane. Do not form closable cone.

7. Three squares around point generate cube.
Four squares lie in plane.
Three pentagons around point generate a dodecahedron (12-faces).
Four pentagons will not fit around a point.
Three hexagons around point lie in plane.

8. There are no further possibilities. The regular polyhedra are:

tetrahedron	(4 faces)	(4 vertices)
cube	(6 faces)	(8 vertices)
octahedron	(8 faces)	(6 vertices)
dodecahedron	(12 faces)	(20 vertices)
icosahedron	(20 faces)	(12 vertices)

9. If you interchange numbers of vertices with numbers of faces, then the cube and octahedron interchange, the dodecahedron and icosahedron interchange, the tetrahedron remains unchanged.

10. If P's vertices are at the centers of Q's faces then P is the dual of Q. If Q is a regular tetrahedron then so is P. The tetrahedron is self-dual.

11. The cube and octahedron are dual; the dodecahedron and icosahedron are dual.

12. It cannot be done with tetrahedra.

13. Form two identical cones with five triangles in each. Join the cones along their rims. The vertices along the rim are common to only 4 faces while the other two are common to 5. Hence the polyhedron is not regular.

ANSWERS

SPHERE-PACKING

1. One. (Note incidentally that two such stacks placed base to base just fit in a regular octahedron.)
2. Twelve. (Four in its own layer, four above and four below.)
3. No. The regular polyhedron would have to be an icosahedron, which has triangular faces. But the polyhedron with vertices at the contact points has two square faces. The four points of contact with the upper layer, for instance, are vertices of a square face.
4. Thirty-five.
5. Twelve. (6 in its own layer, 3 above, 3 below.)
6. This also has square faces, though they are a little harder to see than in the previous case. Alternatively, observe that the polyhedron is symmetric with respect to the plane through the six contact points in the sphere's own layer. Therefore, by symmetry an even number of faces must come together at each of these six points. But the icosahedron has an odd number (five) of faces at each vertex.
7. Yes.
8. Two.
9. One.

F. W. Sinden

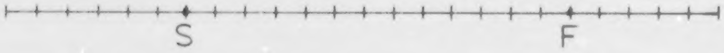
NUMBER THEORETIC GAMES ON A NUMBER LINE

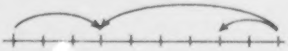
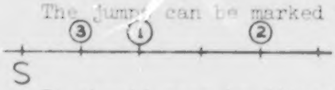
(Grades 3-6)

The basic game described below requires only counting, but it develops into a game giving practice with sums and multiples. The strategic problems raise number theoretic questions which should be approachable by fifth or sixth grade students.

The basic problem is this: given 2 numbers representing "jumps" of so many unit steps (say 4 and 7 unit steps), a starting point and a "target" (say 17 unit steps away), use the given unit jumps to hit the target in as few jumps as possible. Jumps may be made in either direction.

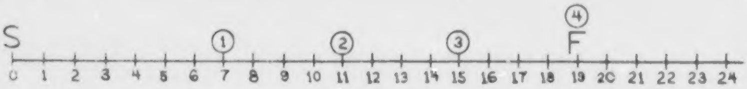
I. At first small numbers can be used and the play can be recorded on a segment (which could be extended, if necessary) marked off in unit subsegments:



The jumps can be marked with arcs  numbered  in order.

It should soon be discovered that no position should be reached more than once in an efficient play. If this is done the points can be marked with objects, say disks, and a jump can consist of removing an object from the segment. The player finishes when he removes the target object and his score is the number of objects he has removed. Low score wins.

II. It should be evident after a little play that a coordinate system will make play easier. The coordinate system could be prescribed or left to the player to select.



The diagram above shows a possible record of play with (4,7) as the jumps.

III. Many extensions of the basic game come readily to mind and several inquiries are suggested by the play; for example:

a) The play can be recorded in shorthand as say, 7, 11, 15, 19. (4).

b) Once the commutativity of jumps is observed an even shorter record can be made; namely

$$1 \cdot 7 + 3 \cdot 4 = 19 \quad (3 + 1 = 4). \quad 3 \cdot 7 - 1 \cdot 4 = 17 \quad (3 + 1 = 4).$$

c) Not every target can be hit with a given pair of steps. Which ones can be hit? Which pairs of steps make a hit possible for a given target distance?

d) The situation above changes if the segment is replaced by a circle. What are the answers in this case?

e) Some targets require backward jumps to be hit at all. Far off targets never seem to. Is this true?

f) Take fractional jumps, say $\frac{3}{4}$ and $\frac{5}{6}$, to hit a target 8 unit steps away. Can any (integral) target be hit with these steps? Can any rational target be hit? etc.

W. Lister

PUZZLES CONCERNING NUMERICAL ODDITIES

Numerical puzzles which catch a students fancy can pay immense dividends with increased skill and understanding of underlying mathematical principles. We shall illustrate this below with a couple of examples. Care must be taken however to present examples of the type given here only to students who recognize them as curiosities. Exhibiting them before the naive may contribute toward increasing the number of misconceptions which they already have.

The first of these can be given students who have learned how to reduce fractions. Put an example, like that below, on the board.

$$\frac{16}{64} = \frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}$$

Several students should object to this, but upon closer inspection will grant that $\frac{16}{64}$ does equal $\frac{1}{4}$. Now show them another:

$$\frac{49}{98} = \frac{4\cancel{9}}{\cancel{9}8} = \frac{4}{8}$$

You may now inform the students that there are other fractions that work the same way and suggest that they may like to find some of them. As an extra caution some other fractions should be investigated which don't have this peculiar property.

The students who do investigate other possibilities will be engaged in much self generated drill and checking for the property in question should strengthen their understanding of equivalent fractions and the common theorems regarding multiplying (dividing) both the dividend and division by the same number. They should also meet with some success with their endeavor too, since there do exist other fractions with this property. Examples of fractions whose numerators and denominators are 2-digit numerals are:

$$\frac{26}{65} \qquad \frac{19}{95}$$

This investigation is quite open ended as students may wish to investigate fractions involving numerals of more than 2-digits.

Another appealing equivalence is illustrated by the statements below.

$$\sqrt{3\frac{3}{8}} = 3\sqrt{\frac{3}{8}} \quad , \quad \sqrt{75\frac{75}{5624}} = 75\sqrt{\frac{75}{5624}}$$

These weird equalities are especially useful in demonstrating the power of a deductive approach over a computational one. Investigating them also gives purposeful experience in dealing with radicals rather than straight drill. It has been expressed several times during our conference that with more effort and imagination, the pages of drill in mathematics text books could be replaced by activities from which students derive the same skills but which are more meaningful, purposeful, enjoyable, and which have greater transfer value to later work. This is not to say we are against "drill" but we are for a more effective drill. In large, drill which stems from an activity important to children at the time and hence is part of the job. This experience does not even seem like drill to the children and has the advantage that it doesn't treat them like a machine which is set for a straight-jacket task.

Now, we shall present an analysis of the two statements above to illustrate the type of "mileage" a teacher can get out of this topic. The first one can be handled in a clean cut computational manner.

$$\sqrt{3\frac{3}{8}} = \sqrt{\frac{27}{8}} = \sqrt{9 \cdot \frac{3}{8}} = \sqrt{9} \cdot \sqrt{\frac{3}{8}} = 3\sqrt{\frac{3}{8}}$$

Hence,

$$\sqrt{3\frac{3}{8}} = 3\sqrt{\frac{3}{8}} \ .$$

The second example would be tedious to handle by the method above and so it might be wise to recall what one does when changing a "mixed number" to an "improper fraction." This in itself is a worthwhile activity for youngsters. In order to do this we shall not carry out our computations which are difficult anyway.

$$\sqrt{75\frac{75}{5624}} = \sqrt{\frac{75(5624) + 75}{5624}}$$

Now by the multiplicative property of one we have,

$$= \sqrt{\frac{75(5624) + 75(1)}{5624}}$$

and by the distributive property of multiplication over addition,

$$= \sqrt{\frac{75(5624 + 1)}{5624}}$$

$$= \sqrt{\frac{75(5625)}{5624}}$$

$$= \sqrt{\frac{75(75^2)}{5624}} .$$

The last step above depends on a well known computation short cut for squaring "2-digit numbers ending in 5." This short cut also has a valuable analysis and can be broadened to multiplying 2-digit numbers with the same tens digit whose units digits have a sum of 10. Continuing:

$$= \sqrt{\frac{75}{5624} \cdot 75^2}$$

$$= \sqrt{\frac{75}{5624}} \cdot \sqrt{75^2}$$

$$= \sqrt{\frac{75}{5624}} \cdot 75$$

$$= 75 \sqrt{\frac{75}{5624}}$$

This derivation reveals a wealth of properties which are accessible to high school or junior high school students. Students may be asked to review the derivation to pick out the properties that were crucial to the examples. They may then be asked to give further examples like those given. The fact that students give examples rather than generalizations is important since the faster kids won't be giving the game away to the others. At the same time they are contributing additional examples upon which the others may draw. After seeing several examples like:

$$\sqrt{2\frac{2}{3}} = 2\sqrt{\frac{2}{3}} \qquad \sqrt{4\frac{4}{15}} = 4\sqrt{\frac{4}{15}}$$

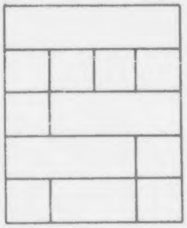
$$\sqrt{5\frac{5}{24}} = 5\sqrt{\frac{5}{24}} \qquad \sqrt{6\frac{6}{35}} = 6\sqrt{\frac{6}{35}}$$

most students will be able to construct their own examples.

H. Wills

TRAINS OF RODS

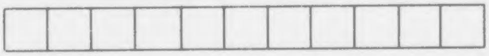
Consider a set of rods similar to Cuisinaire Rods. That is, rods with square bases and lengths which are integral multiples of an edge of a base. Rods of the same length have the same color which is different from the color of a rod of a different length. For young children, the game is to see how many trains they can make of a given length. For instance, here are some of the trains of length 4:



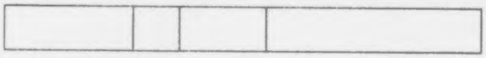
There are others. Different placement of adjacent rods of the same length is not considered a different train whereas different placement of adjacent rods of unequal length is considered different should it yield a new pattern in going from left to right. Hence, there are only 3 more trains possible length 4 other than those illustrated above. This activity presents an elementary problem solving situation to young children as well as gives them much experience in additive decomposition. It also builds up their intuition for commutativity of addition.

For older children this same game can be used as a puzzle. See if they can decide how many trains are possible of length eleven. Once again this problem can be handled at various levels as suggested by the Jumping Game problem. We shall give an analysis here for just the high school student who is equipped with the fact that the number of subsets of an n element set is 2^n.

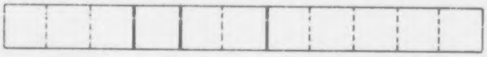
Let us begin by considering a train made up entirely of unit rods. It would look like this:



How many pairs of faces touch "completely" in a train like this? We shall consider those line segments in the picture above which represent pairs or touching faces. That is, we shall consider all vertical line segments above except the end ones. After some investigation we conclude that any train can be determined by taking certain vertical segments and ignoring the others. For example, the train:



is determined by the 3 vertical segments illustrated below.



Hence, we see that for each train there corresponds one and only one subset of the set of vertical lines in question. What set of vertical lines corresponds to the train consisting of a single rod eleven units long? So, we conclude that the number of trains possible having length eleven is 2^{10} which is 1,024. The fact that this number of possibilities would take quite a bit of effort should one attempt to identify them empirically serves a good basis for an appreciation of the application of the mathematics they have learned. Especially if they have occasion to use it more than once in diverse situations. The analysis used here can also be used to solve the Jumping Game problem and both involve the concept of the number of subsets as do many other seemingly unrelated topics experienced by high school students. To mention a few:

The tower of Hanoi.

The chinese rings.

Finding the number of ways 'MATHEMATICS' can be spelled in an array like that below:

M
M A M
M A T A M
M A T H T A M
M A T H E H T A M
M A T H E M E H T A M
M A T H E M A M E H T A M
M A T H E M A T A M E H T A M
M A T H E M A T I T A M E H T A M
M A T H E M A T I C I T A M E H T A M
M A T H E M A T I C S C I T A M E H T A M

The sum of the terms in the n-th row of Pascal's triangle

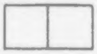
$$\sum_{i=0}^{n} 2^{i} \quad \text{etc.}$$

H. Wills

POLYOMINO PROBLEMS

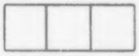
The Polyominoes

Polyominoes are the geometric configuration formed by fitting together a given number of unit squares by joining their edges. The simplest polyomino is the monomino consisting of a single unit square. The next simplest polyomino is the domino formed by joining two unit squares to form a rectangle.

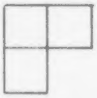


Domino

Trominoes are formed by joining three unit squares. There are two distinct forms of trominoes:



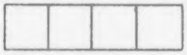
The Straight Tromino



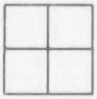
The Right Tromino

Under a suitable rotation, any tromino will assume one of these two forms.

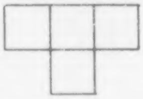
Next in order of complexity are the tetrominoes formed from four unit squares. There are essentially five distinct types of tetrominoes.



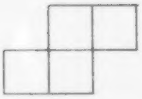
Straight Tetromino



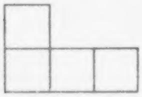
Square Tetromino



T Tetromino



Skew Tetromino

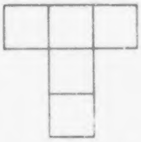


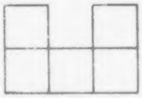
L Tetromino

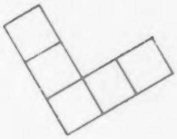
Any tetromino can be put in one of these forms by rotation and/or flipping it over.

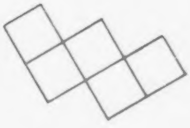
Having seen how the distinct tetrominoes are constructed, it should be an interesting exercise in combinatorial geometry for a student at the appropriate grade level to construct all of the distinct (with respect to rotation and reflection) pentominoes.

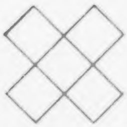
He should discover the following:

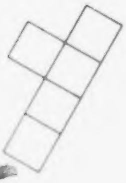


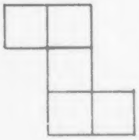


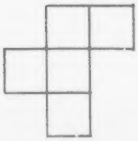


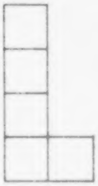


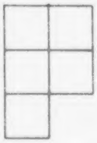


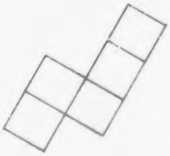












Educational Uses of Polyominoes.

Polyominoes may be used in many ways to give students a feeling for configurational and combinatorial geometry. Furthermore many polyomino problems lead to questions requiring elegant logical analysis and, occasionally, mathematical proof.

The best known of the polyomino problems is the problem of taking a set of polyominoes and fitting them together to form a given pattern. For example, the 12 distinct pentominoes comprise 60 unit squares and hence suggest the problem of fitting them together to form 3×20, 4×15, 5×12, and 6×10 rectangles. Each of these configurations can be made. One of the more difficult problems of this kind, is to divide the set of 12 pentominoes into two sets of six such that each set can be put together to form 5×6 rectangles. While problems of this kind are interesting puzzles and give some experience in configurational geometry, they appear to be of limited educational use.

Covering Problems

A second class of problems is related to the general problem of covering a given configuration of unit squares with replicas of a given polyomino. A common configuration is the ordinary checkerboard or the checkerboard with certain squares removed.

The simplest such problem is that of covering the checkerboard with dominoes. This problem has a trivial solution since the checkerboard can be represented as a 4×4 array of square tetrominoes. Less trivial is the classical problem of covering the checkerboard with two opposite corner squares removed. This problem has no solution. The impossibility can be demonstrated by noting that in the usual red-black coloring of the checkerboard, two opposite corner squares have the same color. So that the remaining squares have more of one color than the other. But any domino covers one red square and one black square so that any configuration covered by dominoes has the same number of black and red squares. Students who are familiar with this solution may find it stimulating to try the tromino covering problems. Since 64 is not divisible by 3, the checkerboard cannot be covered with trominoes. When one square is removed, the number of remaining squares is a multiple of 3 and one can ask of it is possible to cover all but one square with a given tromino. It is an interesting exercise to show that the checkerboard can be covered by right trominoes in such a way that the one uncovered square has any given position on the board. In this case it is instructive to begin with the 2 × 2 board and then treat the 4 × 4 and 8 × 8 boards. The corresponding problem for the straight tromino is more complicated and affords some excellent opportunities for logical analysis. For only certain specified squares can the remaining portion of the board be covered by straight trominoes. The key to this analysis is an extension of the coloring argument used for the domino coverings. However in this case the board must be covered by three colors with the diagonals successively colored, say, red, white, and blue. When the board is so colored, then any tromino placed on the board will cover one white, one red, and one blue square. Hence any configuration covered by trominoes will have an equal number of red, white, and blue squares. By this means the possible configurations with one square uncovered can be determined. It is of course necessary to construct the covering in the cases which are not eliminated by the covering criteria.

R	W	B	R	W	B	R	W
B	R	W	B	R	W	B	R
W	B	R	W	B	R	W	B
R	W	B	R	W	B	R	W
B	R	W	B	R	W	B	R
W	B	R	W	B	R	W	B
R	W	B	R	W	B	R	W
B	R	W	B	R	W	B	R

W	R	B	W	R	B	W	R
R	B	W	R	B	W	R	B
B	W	R	B	W	R	B	W
W	R	B	W	R	B	W	R
R	B	W	R	B	W	R	B
B	W	R	B	W	R	B	W
W	R	B	W	R	B	W	R
R	B	W	R	B	W	R	B

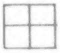
There are 22 Red squares, 21 White squares, and 21 Blue squares. Hence the uncovered square must lie on one of the red diagonals. But the board can just as well be colored using the diagonals running in the perpendicular direction.

The corresponding diagonals in that case intersect the Red diagonals in exactly four symmetric squares. These are precisely the cases for which the solution is possible.

Having considered these cases students may find it challenging to tackle the domino and tromino covering problem for square or rectangular boards of other sizes and to treat the tetromino and pentomino covering problems.

Exclusion Problems

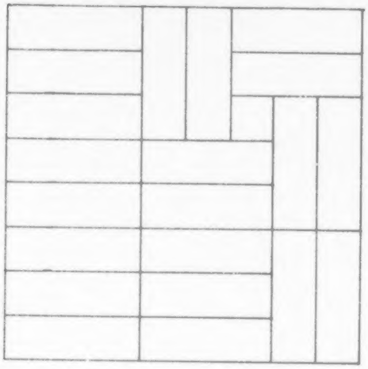
The exclusion problems are concerned with the minimal number of monominoes which can be placed on a checkerboard to make it impossible to place a given polyomino on uncovered squares. For example, if the polyomino is a domino, then placing monominoes on all of the red squares will make it impossible to place a domino on uncovered squares, since any domino so placed will cover one red and one black square. Thus 32 monominoes are sufficient. But since 32 dominoes can cover the board, a monomino must be placed in one of the squares covered by each domino. Hence at least 32 monominoes are required. Thus 32 is the minimal number.

In the case of trominoes, it is clear that 32 monominoes suffice to exclude right trominoes since all dominoes can be excluded. But 32 is the smallest number since every 2 × 2 array  must have at least two monominoes if the right trominoes are to be excluded.

The following diagram shows that 21 monominoes are sufficient to exclude straight trominoes.

	×			×			×
		×			×		
×			×			×	
	×			×			×
		×			×		
×			×			×	
	×			×			×
		×			×		

In order to see that 21 are required, note that 21 trominoes can be placed on a checkerboard as follows:



Since a monomino must be placed on one of the squares covered by each tromino, at least 21 are required. Hence 21 is the minimal number of monominoes required to exclude straight trominoes. It should be noted that there is a close relationship between the tromino covering problem and the tromino exclusion problem.

The exclusion problem for tetrominoes and pentominoes can be attacked by similar methods, though the solutions are more complicated. The exclusions problem for X, F, Z are particularly challenging. The minimal number of monominoes required for exclusion of the various pentominoes are given in the following table:

T	U	V	W	X	Y	Z	F	L	I	P	N
14	16	16	15	10	16	14	14	16	12	16	16

The examples given here are just a selection of polyomino problems which are of educational interest. Many more problems are treated in S. Golomb's book, *Polyominoes*, Charles Scribner's Sons, New York, 1965.

R. P. Dilworth

POWERS AND PRIMES

Select a prime number. Divide each power of 2 by that prime and record the remainder in the correct cell. Continue this activity for each prime.

	2	3	5	7	11	13	17	19	23	29
2										
4										
8										
16										
32										
64										
128										
256										
512										
1024										
2048										

This chart provides many opportunities for students to recognize patterns and make discoveries which are related to rich areas in mathematics.

A teacher might start out by listing the powers of 2 down the chalk board in order and ask for students to continue giving entries according to the pattern the teacher has in mind. A similar activity can be performed for the primes across the top.

At this point we will diverge briefly from the major topic at hand to record an activity, brought out in our discussions, which relates to prime numbers.

The children are given some type of counters--pennies, pebbles, checkers, chips, pegs, etc.--and asked to see how many different rectangular arrays they can form with a given number of counters. Any n by m array is considered to be the same as any other n by m array as well as the same as any m by n array. Each child should conduct his own experiments and summarize his findings in a table like this one:

Number of Tokens	1	2	3	4	5	6	7	8	9	10	11 ...
Number of Rectangular Arrays	1	1	1	2	1	2	1	2	2	2	1 ...

Students may now be told that those numbers for which there is only one rectangular array are the prime numbers (except for the number one which is or isn't depending upon the definition or particular author). This activity could be carried on along with getting kids to discover the pattern across the top of the chart. Upon looking at the table generated from investigating rectangular arrays, a student might be asked questions similar to the following.

Is there a number for which more than two rectangular arrays may be formed?

How many rectangles can be formed with six tokens? How many factors has six?

How many rectangles can be formed with nine tokens? How many factors has nine?

Find a number between 30 and 100 which has exactly 3 factors. How many such numbers are there?

The activity of forming rectangles can also be lowered down into the primary grades to teach multiplication. cf. The Advanced Montessori Method Vol. II by Maria Montessori.

In filling the powers of two-prime chart, column by column students recogniz many patterns and are rewarded by this recognition since following the patterns greatly speeds their task. However, soon even the patterns become difficult to construct. This motivates students to make further discoveries which are immediately rewarding.

A very satisfying discovery is to abandon the set rules to divide a prime into the various powers of 2 and merely begin to double the previous remainder and record the excess above any multiple of the prime in question. This serves as a good introduction or application to modular arithmetic as well as some of the basic theorems on congruence such as Fermat's Little Theorem:

When p is prime and a is not divisible by

$$a^{p-1} \equiv 1 \pmod{p}$$

H. Wills

A PROBLEM IN DIVIDING FRACTIONS

Take the first five prime numbers -- 2, 3, 5, 7, 11 -- and write them with four division signs of unequal length in the following way:

$$\cfrac{2}{\cfrac{\frac{3}{5}}{\frac{7}{11}}}$$

The value of this expression is given by the rules that (a) the longest division sign is applied first, and (b) this rule is repeated until a simple fraction is obtained. Successively, one gets

$$2 \div \cfrac{\frac{3}{5}}{\frac{7}{11}} = 2 \times \frac{7}{11} \div \frac{3}{5} = \frac{2 \times 5 \times 7}{3 \times 11}$$

The final result need not be multiplied out.

By putting the division signs in a different sequence, a different value may be obtained. Thus,

$$\cfrac{2}{\cfrac{3}{\cfrac{5}{\cfrac{7}{11}}}} = 2 \div \cfrac{3}{\cfrac{5}{\cfrac{7}{11}}} = 2 \times \cfrac{5}{\cfrac{7}{11}} \div 3$$

$$= 2 \times 5 \div \frac{7}{11} \div 3 = \frac{2 \times 5 \times 11}{3 \times 7} .$$

How many different arrangements of signs are possible? Do they all give different answers?

Try

$$\cfrac{2}{\cfrac{\frac{3}{5}}{\frac{7}{11}}}$$

How many distinct values are possible?

The primes are used so that all repetitions occur only as a result of equivalent arrangements of signs. The problem provides examples of non-associativity in arithmetic operations, gives practice in handling division by multiplying reciprocals, tests accuracy in handling compound fractions, and is challenging from the combinatory point of view. For this latter aspect, have the students look at the same problem with 2, 3, or 4 division signs and try to find the rule for the number of distinct values obtained in the general case.

W. Jacobs

LOGIC PUZZLES

"I don't like spinach and I'm glad I don't
because if I did, I'd eat it and I hate it."

Logic puzzles can be useful in overcoming the purely linguistic difficulty of expressing things accurately in the ambiguous, redundant, variable medium of English. The syllogism puzzles of Lewis Carroll are especially good for this because the absurdity of their content serves to emphasize that only the form is of any importance in logic. These puzzles can be found, arranged in a graded sequence, in "Logical Nonsense", Putnam, 1934, pp. 505-546. For use with modern children, they need to be sifted and modified somewhat. In particular it might be a good idea to start with equivalent forms of single statements (which Lewis Carroll doesn't have) before going on to multi-statement puzzles. Following are a few examples (variants of L. C. assertions). For teaching purposes one would want many dozens of these to be worked first by common sense then by more systematic methods.

A. Equivalent forms of single statements. Do the two statements say the same thing or different things? (Answers below)

1. No large birds live on honey. If a bird lives on honey, it is not large.
2. No one, who forgets a promise, fails to do mischief. Anyone who forgets a promise does mischief.
3. All, who are anxious to learn, work hard. All, who work hard, are anxious to learn.
4. Prudent travellers carry plenty of small change.
   Travellers without small change are imprudent.
5. No child is healthy who takes no exercise.
   Children who exercise are healthy.
6. Some elderly ladies are talkative
   Some talkative persons are elderly ladies.
7. Nobody, who really appreciates Beethoven, fails to keep silent when the Moonlight Sonata is being played.
   People, who keep silent when the Moonlight Sonata is being played, really appreciate Beethoven.

8. No drug is useful in a toothache, unless it relieves pain.
A drug that relieves pain is useful in a toothache.

9. None but a hop-scotch player knows real happiness.
All who know real happiness are hop-scotch players.

10. Whenever I do not take my umbrella it rains.
Whenever it does not rain, I take my umbrella.

Answers:

1. same
2. same
3. different
4. same
5. different
6. same
7. different
8. different
9. same
10. same

After some practice with these, one can go on to puzzles involving two or more statements such as these:

B. Decide whether the conclusion follows from the two statements. (Answer below)

11. "I saw it in a newspaper."
"All newspapers lie."

Conclusion: It was a lie.

12. Every eagle can fly.
Some pigs canhot fly.

Conclusion: Some pigs are not eagles.

13. All, who are anxious to learn, work hard.
Some of these boys work hard.

Conclusion: Some of these boys are anxious to learn.

C. In each of the following examples, draw a conclusion, if possible.

14. All ducks waddle
Nothing that waddles is graceful.

15. Some unkind remarks are annoying.
No critical remarks are kind.

16. Canaries, that do not sing loud, are unhappy.
No well-fed canary fails to sing loud.

17. All puddings are nice.
This dish is a pudding.
No nice things are wholesome.

18. Nobody who really appreciates Beethoven fails to keep silent while the Moonlight Sonata is being played.
Guinea-pigs are hopelessly ignorant of music.
No one who is hopelessly ignorant of music ever keeps silent while the Moonlight Sonata is being played.

Answers:

11. Conclusion does not follow.
("All newspapers lie" is taken to mean "All newspapers sometimes lie." It could conceivably mean "All newspapers always lie." Such ambiguities should be avoided at first, later discussed.)
12. Conclusion follows.
13. Conclusion does not follow.
14. All ducks are ungraceful.
15. No conclusion.
16. Some ill-fed canaries are unhappy. (Lewis Carroll does not admit the possibility that the set of soft-singing canaries is empty. By modern standards one should admit this possibility. The important thing, though, is to be clear about the rules of the game. One implicit rule is that "some" means "one or more." Another implicit rule is that everything has at most two values: Canaries are either well-fed or ill-fed. Indifferently fed canaries do not exist.)
17. This dish is not wholesome.
18. Guinea-pigs do not really appreciate Beethoven.

These are a few examples to illustrate the tone and difficulty of these puzzles. For teaching purposes one should have dozens of them. After doing a great many by common sense one could introduce the set interpretation together with the usual visual aids (Venn diagrams). This should give the pupils quite a sense of power and enable them to go on to harder puzzles involving many statements. Several such are given in Logical Nonsense (loc. cit) including one with twenty statements. Other multi-statement logic puzzles can be found in the standard puzzle books, but without the Carrollean whimsy they often seem stodgy and artificial.

Because of its importance in mathematics, one should also practice a lot with the if .... then form. In particular the children should learn the work-horse rules about converses and negations.

Examples: In each set which statements are equivalent?

1. No child is healthy who takes no exercise.
   If a child takes exercise, then he is healthy.
   If a child is not healthy, then he takes no exercise.

2. No country, that has been explored, is infested with dragons.
   If a country is infested with dragons, then it has not been explored.
   If a country has not been explored then it is infested with dragons.
   If a country has been explored, then it is not infested with dragons.

One can, of course, go on to more formal logic systems, Boolean Algebra, switching circuits, etc., but this is really a different subject, mathematics rather than linguistics. Puzzles of the Lewis Carroll kind are primarily linguistic: they concern equivalences between forms of English sentences. Skill in handling these equivalences is essential if one is to talk about mathematics in the English language.

Following is another logic puzzle that I happen to know and like.

The District Attorney

This is basically a puzzle, but it can be made into a guessing game by having someone in the know act out the part of the D.A. Even when people see the D.A. in action they often have difficulty divining his strategy, which can be made more mysterious by using different but logically equivalent questions each time through.

One of three suspects is guilty. The innocent ones can be counted on to tell the truth, but the guilty one may or may not tell the truth. The D.A. is to find out who is guilty with just two yes-no questions. At first glance it may seem impossible to get any information at all, but the D.A. does it, e.g., as follows:

D.A.: (to suspect A) All right A, either you or B did it, right?

A: No

D.A.: Then I can only conclude that C did it. (to suspect B): Will you confirm that C did it?

B: Yes

D.A.: Of course. C did it.

Alternative scenario:

D.A.: (to C) You look honest. Now tell me the truth. Did A do it?

C: No.

D.A.: (to A) If C were guilty he would have said, "yes" in order to frame you. Therefore it must be B. B did it, right?

A: No

D.A.: All right C. The jig is up. Honesty will get you nowhere.

C: But what if I had lied?

D.A.: I would have found out just the same. Dishonesty is no better than honesty.

C: You can't win.

D.A.: True.

False Proof by Induction

At the high school level after students have had experience with mathematical induction they should try this one:

Theorem: In any set of marbles all the marbles are the same color.

Proof (by induction) Let n be the number of marbles in the set. The theorem is certainly true if $n = 1$.

Induction: Suppose the theorem is true for n. Then it is surely true for $n + 1$, for if I remove any marble from a set of $n + 1$, the remaining ones, constituting a set of n, must all be the same color. Since this is true no matter which marble I remove, all $n + 1$ marbles must be the same color.

F. W. Sinden

DOES THE ORDER MAKE A DIFFERENCE?

1. In a single purchase, you are offered 3 successive discounts of 20 percent, 10 percent, and 5 percent, and can take them in any order that you wish. What order would you choose?
2. Arthur and Bob start a game with equal amounts of money. Arthur loses the first game and pays Bob 20 percent of his money. Then Bob loses the second game and pays Arthur 20 percent of the amount Bob has. Do they again have equal amounts of money?
3. Mr. Jones has two houses, which cost him the same amount. He sells one house at a 10 percent profit to Mr. Allen, who resells it to Mr. Baker at a 10 percent loss. Mr. Jones sells his second house at a 10 percent profit to Mr. Dahl. Which paid more, Mr. Baker or Mr. Dahl?

IDENTIFICATION PROBLEMS

The class of tricks based on what is generally called Gergone's Pile Problem provide an example of the power of numerical coding and an application of place system ideas. A typical version is the following: The "magician", A, asks B to deal 27 distinguishable cards in 3 equal piles, to select a card mentally and to announce which pile it is in. A then tells B to reassemble the cards by placing one pile on top of the other in any order. A notes the order in which the piles are assembled. After two repetitions of this cycle A tells B the position of his card in the reassembled deck.

If this trick is introduced to a fourth or fifth grade class in its 2 pile, 4 card form an analysis by the class should be possible and can be made to relate to binary numeration. The binary form and the general case can be investigated at an appropriate later time.

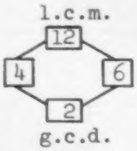
A complete discussion of several variations of this trick can be found in Martin Gardner's Mathematics, Magic, and Mystery (Dover, 1956).

W. Jacobs

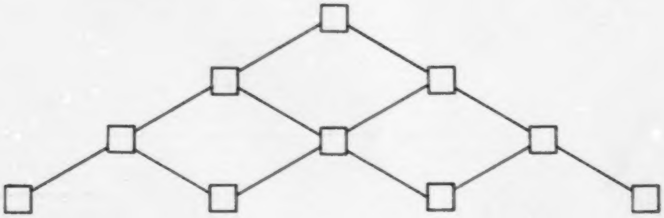
A GAME WITH FACTORS AND MULTIPLES
(Grades 4-6)

The type of game outlined below depends solely on the multiplicative aspect of the positive integers. The objective is to explore factor relationships and incidentally to reinforce direct recall of the multiplication facts basic to computation.

The game is based on the g.c.d. and l.c.m. diagram exemplified by



To make a game consider a triangular array of say 10 boxes.

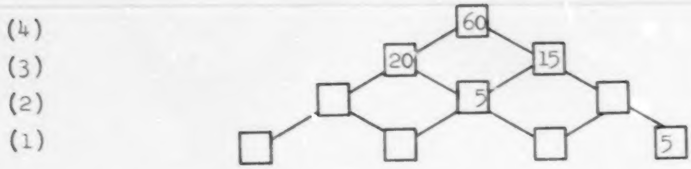


The game is played by 2, or perhaps 3, players, one of whom might be the teacher. Player A begins by filling in the top box. The other "players" in turn fill in the other boxes. In a simple version the only rule governing permissible entries is that the number put in the top box may never be used again.

The Play

After an entry is made any other player may challenge the player making it to show that the configuration is part of a "factor diagram". If the latter succeeds he wins (or scores so many points). If he fails he loses (or his challenger scores so many points). At any time a player believes his play makes the configuration part of a unique factor diagram he wins (or scores) upon convincing the other players of his assertion.

Here is a play.



A	60(4)		20(3)		15(3)
B		5(1)		5(2)	

This last play can be challenged.

Additional rules for admissible entries may be added. It may be required: for example, that

i) The entries in row (1) must be relatively prime in pairs, or

ii) The entries in row (1) must be primes or 1, or

iii) The number 1 may not be used.

W. Lister

A STOCHASTIC NUMBER LINE GAME

There are many commercial games which are essentially stochastic number line games. Typically, in these games the players move playing pieces along a path on the playing board, the length of the move being determined by a throw of a die or a pair of die. The interest in such games is usually supplied by a system of rewards and penalties for stopping a specific position along the path. While these games may interest children - and sometimes adults - the games are not particularly instructive since the play is purely a chance phenomena and thus an understanding of the probabilistic principles underlying the game can in no way aid the perceptive player. In order to make the game instructive, particularly with regard to probability, it should be possible for the perceptive player to make use of the principles of probability to improve the strategy of his play. We give an example of such a game.

They play will consist of moves on the number line although any discrete linearly ordered representation will do. Any number of players may play and all start at the point labeled 0. The players in turn throw a pair of dice and move either forward or backward a distance equal to the number of dots showing on the two dice. It is this feature of being able to choose either a forward or backward direction for the move, that enables a player to improve his chances of winning. The object of the game is to reach the points numbered 6, 12, 18, etc., in that order. Thus, a player at his turn will throw the dice, moving forward or backward until he reaches 6, and then will proceed to 12, 18, etc. If he reaches 12 before reaching 6, this does not count and he must again reach 12 after having landed on 6. The first player to reach a given target, say 30 or 60, wins.

A perceptive player can clearly make use of knowledge about the distribution of the number of dots showing when to dice are thrown to improve his play. For example, if he is at 7 and throws a 4, it is clearly advantageous for him to move forward rather than backward since he will have a much better chance of throwing a 5 on his next turn than he will of throwing a 3.

Note that if a single die is thrown then the best strategy is much simpler, for at each turn the player should always move in the direction of the position which he is trying to reach. It is instructive to analyze the best strategy for play if a die is thrown which has two 1's and four 2's on its faces, or if it has one 1, two 2's, and three 3's.

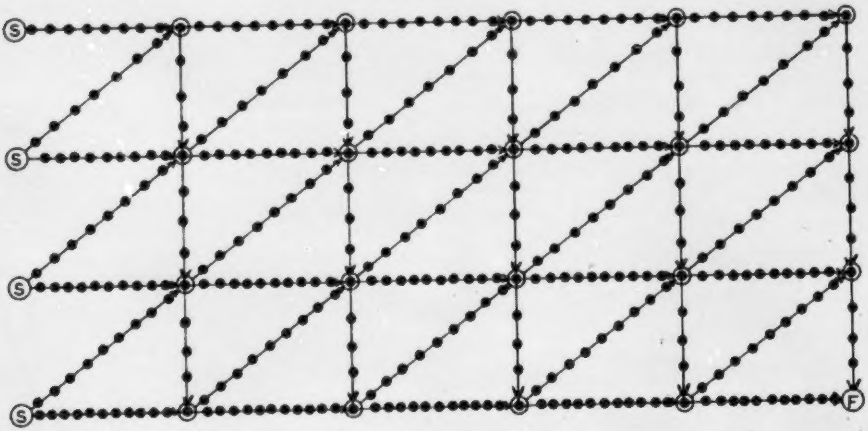
It is clear that games of this type give students considerable incentive to learn and use probability principles which will improve their chances of winning. With suitable choices of the basic distribution much of basic probability can be learned in this way. Furthermore, the kind of probabilistic reasoning required is very similar to the kind needed to make effective probability judgments in ordinary life situations.

R. P. Dilworth

GAMES ON DIRECTED GRAPHS

Stochastic games on the number line are a special case of stochastic games on directed graphs. Again there exist a number of commercial games which have the players move through a network, each move being determined by the throw of a die or a pair of dice. These games are likewise not particularly instructive since the play is usually a purely chance affair. However, games on directed graphs can easily be constructed in which an understanding of the underlying probability can be used to improve the strategy of play. Such games are potentially instructive particularly with regard to basic principles of probability. We give an example of a comparatively simple stochastic game on directed graph.

The playing board has a diagram as follows:



The game is played by four players who start in the four circular positions label S. They may draw lots to determine who starts in each position. They play in the order I, II, III, IV and, in his turn, each player throws a pair of dice to determine how far he will move along the directed graph. He may move along any path consistent with the arrows. His objective is to reach the finish position F in the fewest number of moves. The player first passing through a junction secures it for himself and other players may not follow paths passing through that junction.

The first player reaching F makes a score of 60 - n where n is the number of dots he has traversed in reaching the finish. All other players score zero. Note that the first player to reach F may still get a negative score.

This game clearly has a great many more strategic possibilities than the previous number line game. For example, player IV gains nothing by heading to the finish position directly since his score is then zero, but by taking a path higher into the graph he increases his chances of being cut off. Likewise, player I must decide whether he should take a path around the outside where he can make a good score but runs the risk of being cut off of taking a path into the graph and doing some cutting off himself. Perhaps the numbers of dots between junctions will have to be altered if a wide variety of strategies is to be obtained. Certainly, the distribution of the possible steps in each play must be used in estimating the effectiveness of a given strategy.

R. P. Dilworth

ELEMENTARY THEORY OF AREAS

There is a good deal of puzzle value in dissection problems and such experience should certainly help develop spatial geometric intuition especially the notions of Euclidian motions or congruences.

A pair of polygonal regions A and B are "equivalent" if A can be dissected into polygonal regions which can in turn be reassembled to form B. For example, in FIG. 1, A is equivalent to B, B to C and A to C.

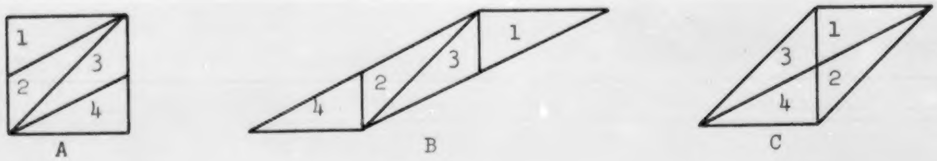


FIGURE 1

It is clear that any pair of equivalent polygonal regions have the same area; but the elegant thing is that the converse is also true: thus: "Every pair of polygonal regions with the same area are equivalent." It is this theorem which is illustrated here.

1. Show that the two parallelogram A and B, (see FIG. 2), which have the same heights and bases, are equivalent.

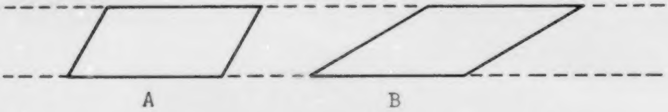
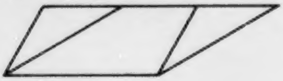
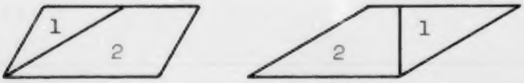


FIGURE 2

(Answer) superimpose bases



and a proper dissection becomes clearer:



2. Show that the two parallelograms A and B in FIG. 3 (in which they are illustrated with their bases superimposed) are equivalent.

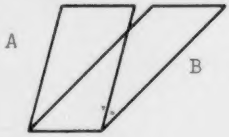
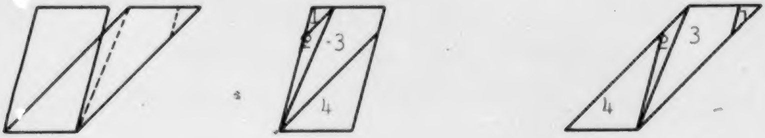


FIGURE 3

(Answer) This is similar to problem 1 but requires an extra step. The The following illustration gives the idea:



3. Show that the triangular region A is equivalent to the rectangular region B. (B has half the height of A).

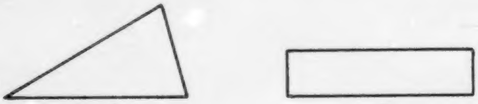
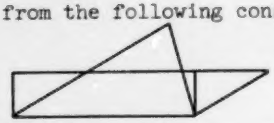
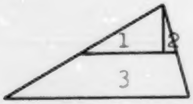


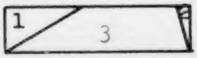
FIGURE 4 A B

(Answer): from the following construction

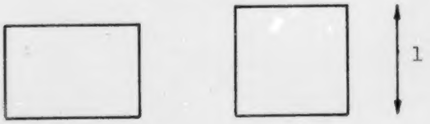


one obtains the dissection :



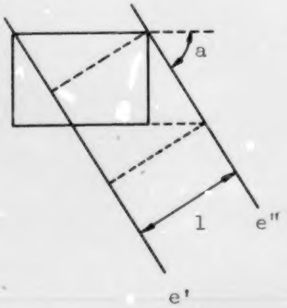


4. Show that the rectangles A and B in FIG. 5 bound equivalent regions. The height of A is less than one and its length is greater than one. The height of B is equal to one and (since the areas bounded by A and B are equal) the length of B must be the common area of A and B.



FIGURES A B

(Answer): make use of following construction



choose angle a so that distance between e' and e" is one.

5. Using the devices indicated in problems 1, 2, 3, 4 one can show that any triangle is equivalent to a rectangle of height one. It seems reasonable (and it can in fact be proved) that any polygon can be dissected into triangles; for example see FIG. 6.

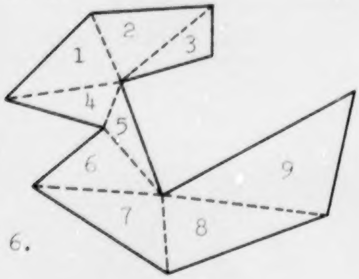
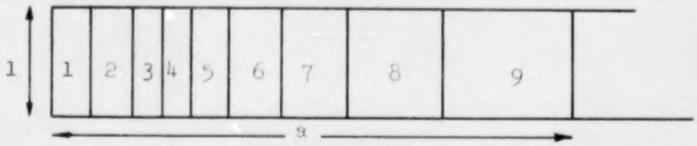


FIGURE 6.

Using the result on triangles one can dissect each of the triangles and rearrange into a rectangle of height one; thus



This not only computes the area "a" of the original polygonal region but also goes a long way toward showing the general result mentioned on page 1.

C. J. Titus

MAP COLORING PROBLEMS

In the following we are concerned with "maps"; for example,

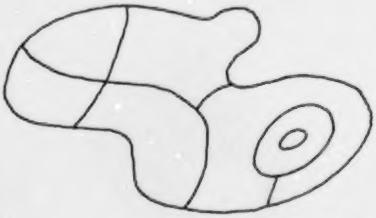


FIGURE 1;

and we are also concerned with the coloring of maps in a special way. We will say a map is "correctly colored" if no pair of bordering countries have the same color; for example, the following is a correct coloring" of the map in FIGURE 1:

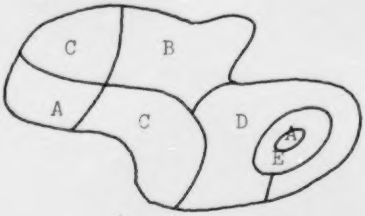
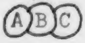


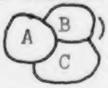
FIGURE 2;

1. Can you "correctly color" the map in FIGURE 1 with fewer than the five colors used in FIGURE 2?

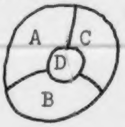
2. What is the least number of colors with which on can correctly color the map in FIGURE 1? (ans: 3)

3. Draw a map which can be correctly colored with two colors. (sample answer: )

4. Draw a map that can be correctly colored with 3 colors but that cannot be correctly colored with 2 colors. (sample answer: the map in FIGURE 2).

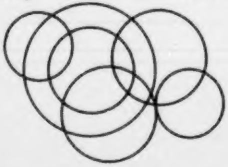
5. How many countries did you have in the map in problem 4? Can you achieve the same result with fewer countries? What is the smallest number of countries one can have in a map and still have the same result? (answer: 3; sample map 

)

6. Draw a map that can be correctly colored with 4 colors but that cannot be correctly colored with 3 colors. Can you achieve the same result with fewer countries? What is the smallest number of countries one can have in a map and still have the same result? (answer: 4; sample map 

)

7. Make a map using only circles; for example,



Correctly color your map with two colors. Notice that you cannot find such a "circle map" that requires more than 2 colors. Can you see why it is that every circle map can be colored with two colors? (sample answer: write a number in each country which is the number of circles in which that country is contained. Color the even numbered countries one color and the odd numbered countries the other color.

C. J. Titus

A COMBINATORIAL PROBLEM SOLVED BY GEOMETRY

Consider the eight (2^3) triplets of 0's and 1's:

000
001
010
011
100
101
110
111

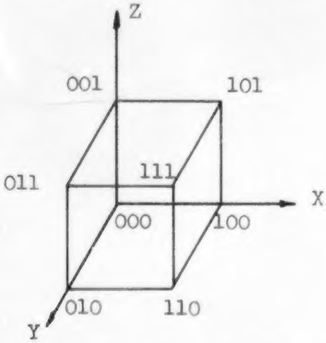
Can you arrange these in a cycle so that neighboring triplets differ in only one place? This problem arises in computer design where the 0's and 1's represent states of on-off devices (e.q. switches, lights, cores). A triplet of such devices is supposed to run through all of its eight states over and over again. For engineering reasons it may be awkward to switch two or more of the devices simultaneously. Therefore one would like to find a sequence that requires switching only one at each step.

This looks like a hard problem. The number of possible cycles is enormous.

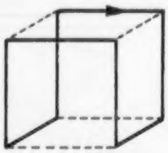
$$\frac{81}{8} = 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7 = 5040$$

(Division by 8 because cyclic permutations indistinguishable)
It turns out, though, that a geometric interpretation makes the problem easy. Consider the triplets of 0's and 1's to be coordinates of points in 3 space.

The eight points represented by the triplets lie at the vertices of a cube.



Neighboring vertices (those joined by an edge) differ in exactly one coordinate. The problem then, is to find a closed path along the edges of the cube which passes through each vertex once. Such a path (Hamilton line) is easy to find:



Tracing out this path, one gets the solution:

000
001
101
100
110
111
011
010

All other Hamilton lines are rotations of the one shown above.

This problem offers an excellent opportunity to discuss higher dimensional geometry. To solve the problem with quadruples of 0's and 1's one considers Hamilton lines on a 4 - cube. To show what a 4 - cube is, one can display cubes of 0, 1, 2 and 3 dimensions, note the induction principle, use it to get the 4 - cube.

0 - cube

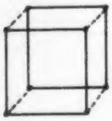
1 - cube

(2 0-cubes joined by a segment)

2 - cube

(2 1-cubes with corresponding vertices joined)

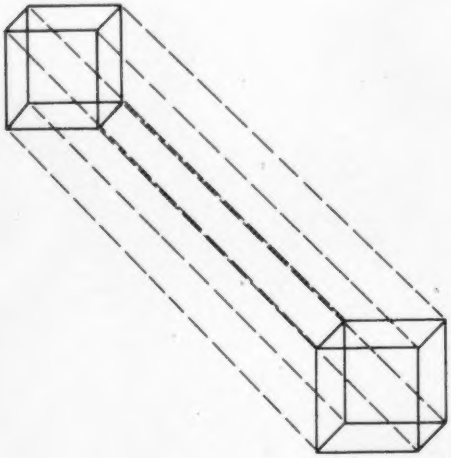
3 - cube



(2 2-cubes with corresponding vertices joined)

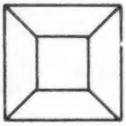
4 - cube

(2 3-cubes with corresponding vertices joined)



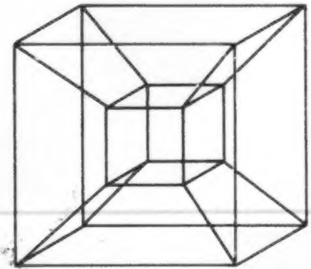
The drawing is neater if you put one cube inside the other:

3 - cube:



(one 2-cube inside the other)

4 - cube:



(one 3 - cube inside the other)

The construction of the Hamilton line can be generalized. Observe that the Hamilton line on a 3-cube first traverses one of the 2-cubes, then jumps to the other 2-cube, traverses it, jumps back to starting point. On the 4-cube: First traverse a 3-cube, jump to the other 3-cube, traverse it, jump back to starting point.

F. W. Sinden